

STRESS-ENERGY TENSOR AND ANGULAR MOMENTUM FOR DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.26.

The stress-energy tensor appears in the definition of Noether's theorem and is defined in L&P's equation 2.44 as

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \partial^\nu \Phi^A - g^{\mu\nu} \mathcal{L} \quad (1)$$

where there is a sum over the index A , which labels the independent fields Φ^A . \mathcal{L} is the Lagrangian density and $g^{\mu\nu}$ is the metric tensor in flat space-time. The stress-energy tensor is used to define a current

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu \quad (2)$$

For the Dirac field, L&P use the Lagrangian

$$\mathcal{L} = \bar{\psi} (i\partial - m) \psi \quad (3)$$

If the action is invariant under some infinitesimal transformation which gives rise to variations $\delta \Phi^A$ in the fields and δx_ν in the coordinates, then the corresponding current is conserved, so that

$$\partial_\mu J^\mu = 0 \quad (4)$$

Starting with L&P's equation 2.56, there is a discussion of the conserved current if the Lagrangian is invariant under a Lorentz transformation. The variation in the field is given by their equation 2.57 as

$$\delta \Phi^A(x) = \frac{1}{2} \omega_{\lambda\rho} \left(\Sigma^{\lambda\rho} \right)_B^A \Phi^B \quad (5)$$

where $\omega_{\lambda\rho}$ are the components (independent of x) of the Lorentz transformation. The quantity $\Sigma^{\lambda\rho}$ is stated to be a 'spin matrix', although no other mention is made of it, nor are any examples given of what it is. I have to confess that I don't understand this, nor do I understand what the indexes A

and B mean with regard to $\Sigma^{\lambda\rho}$. In any case, they go on to show that as a result of Noether's theorem, we can state

$$\partial_\mu \mathcal{M}^{\mu\lambda\rho} = 0 \quad (6)$$

where

$$\mathcal{M}^{\mu\lambda\rho} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \left(\Sigma^{\lambda\rho} \right)_B^A \Phi^B - \left(T^{\mu\lambda} x^\rho - T^{\mu\rho} x^\lambda \right) \quad (7)$$

and the corresponding conserved charge is

$$J^{\lambda\rho} = \int d^3x \mathcal{M}^{0\lambda\rho} \quad (8)$$

The problem in L&P's exercise 4.26 is to calculate $\mathcal{M}^{\mu\lambda\rho}$ and hence $J^{\lambda\rho}$ for the Dirac lagrangian 3. The stress-energy tensor is fairly straightforward. Since only derivatives with respect to ψ occur in 3, we have

$$T^{\mu\nu} = \bar{\psi} i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \mathcal{L} \quad (9)$$

To calculate $\mathcal{M}^{\mu\lambda\rho}$, however, we need to know what $\Sigma^{\lambda\rho}$ is. L&P state just before equation 4.72 that $\frac{1}{2}\sigma_{\mu\nu}$ is the spin operator for fermions, but if we use this as $\Sigma_{\mu\nu}$, we don't get the answer given in the back of the book. It appears that we need to use

$$\Sigma^{\lambda\rho} = -\frac{i}{2}\sigma^{\lambda\rho} \quad (10)$$

If anyone knows why this is so, or can give an explanation of exactly what $\Sigma^{\lambda\rho}$ is, I'd be most grateful.

In any case, using 10, we have from 7

$$\mathcal{M}^{\mu\lambda\rho} = \bar{\psi} \gamma^\mu \frac{1}{2} \sigma^{\lambda\rho} \psi - \left(T^{\mu\lambda} x^\rho - T^{\mu\rho} x^\lambda \right) \quad (11)$$

To calculate $J^{\lambda\rho}$ from 8, we need $T^{0\nu}$ which we get from 9:

$$T^{0\nu} = \bar{\psi} i \gamma^0 \partial^\nu \psi - g^{0\nu} \mathcal{L} \quad (12)$$

$$= i \psi^\dagger \partial^\nu \psi - g^{0\nu} \mathcal{L} \quad (13)$$

using the properties $\bar{\psi} = \psi^\dagger \gamma^0$ and $(\gamma^0)^2 = 1$. We therefore have

$$\mathcal{M}^{0\lambda\rho} = \frac{1}{2} \psi^\dagger \sigma^{\lambda\rho} \psi - i \psi^\dagger \partial^\lambda \psi x^\rho + g^{0\lambda} \mathcal{L} x^\rho + \quad (14)$$

$$i \psi^\dagger \partial^\rho \psi x^\lambda - g^{0\rho} \mathcal{L} x^\lambda \quad (15)$$

To calculate $[\psi, J_{\mu\nu}]$ we need the results

$$\left\{ \psi(t, \mathbf{x}), \psi^\dagger(t, \mathbf{y}) \right\} = \delta^3(\mathbf{x} - \mathbf{y}) \quad (16)$$

$$\left\{ \psi(t, \mathbf{x}), \psi(t, \mathbf{y}) \right\} = 0 \quad (17)$$

[Remember that this is actually a matrix equation, with components $\left\{ \psi_a(t, \mathbf{x}), \psi_b^\dagger(t, \mathbf{y}) \right\} = \delta_{ab} \delta^3(\mathbf{x} - \mathbf{y})$.]

We can apply 16 to the various terms in 14 when placed in the integral 8. We'll consider each term separately.

First, we look at

$$\left[\psi(y), \int d^3x \frac{1}{2} \psi^\dagger(x) \sigma^{\lambda\rho} \psi(x) \right] \quad (18)$$

To simplify the notation, I'll indicate dependence on x or y by a subscript rather than parentheses, so that $\psi_y \equiv \psi(y)$ and so on. We have

$$\left[\psi(y), \int d^3x \frac{1}{2} \psi^\dagger(x) \sigma^{\lambda\rho} \psi(x) \right] = \frac{1}{2} \int d^3x \left[\psi_y \psi_x^\dagger \sigma^{\lambda\rho} \psi_x - \psi_x^\dagger \sigma^{\lambda\rho} \psi_x \psi_y \right] \quad (19)$$

$$= \frac{1}{2} \int d^3x \left[\delta^3(\mathbf{x} - \mathbf{y}) \sigma^{\lambda\rho} \psi_x - \right] \quad (20)$$

$$\psi_x^\dagger \psi_y \sigma^{\lambda\rho} \psi_x - \psi_x^\dagger \sigma^{\lambda\rho} \psi_x \psi_y \right] \quad (21)$$

$$= \frac{1}{2} \sigma^{\lambda\rho} \psi_y + \int d^3x \left[\psi_x^\dagger \sigma^{\lambda\rho} \psi_x \psi_y - \psi_x^\dagger \sigma^{\lambda\rho} \psi_x \psi_y \right] \quad (22)$$

$$= \frac{1}{2} \sigma^{\lambda\rho} \psi(y) \quad (23)$$

Next, we look at

$$\left[\psi_y, \int d^3x i\psi_x^\dagger \partial^\lambda \psi_x x^\rho \right] = i \int d^3x \left(\psi_y \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y \right) \quad (24)$$

$$= i \int d^3x \left(\delta^3(\mathbf{x} - \mathbf{y}) \partial_x^\lambda \psi_x x^\rho - \right. \quad (25)$$

$$\left. \psi_x^\dagger \psi_y \partial_x^\lambda \psi_x x^\rho - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y \right) \quad (26)$$

$$= i \partial_y^\lambda \psi_y y^\rho + \int d^3x \left(\psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y \right) \quad (27)$$

$$= i \partial_y^\lambda \psi(y) y^\rho \quad (28)$$

Note that, in these equations, the derivatives are with respect to the integration variable x and thus do not affect ψ_y , so it is valid to anticommute ψ_y and $\partial^\lambda \psi_x$.

Finally we need to consider

$$\left[\psi_y, \int d^3x g^{0\lambda} \mathcal{L} x^\rho \right] = g^{0\lambda} \left[\psi_y, \int d^3x \bar{\psi}_x (i\partial_x - m) \psi_x x^\rho \right] \quad (29)$$

$$= g^{0\lambda} \left[\psi_y, \int d^3x \psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho \right] \quad (30)$$

$$= g^{0\lambda} \int d^3x \left(\psi_y \psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho - \right. \quad (31)$$

$$\left. \psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho \psi_y \right) \quad (32)$$

$$= g^{0\lambda} \left\{ \int d^3x \delta^3(\mathbf{x} - \mathbf{y}) \gamma^0 (i\partial_x - m) \psi_x x^\rho - \right. \quad (33)$$

$$\left. \int d^3x \left[\psi_x^\dagger \gamma^0 \psi_y (i\partial_x - m) \psi_x x^\rho - \psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho \psi_y \right] \right\} \quad (34)$$

$$= g^{0\lambda} \gamma^0 (i\partial_y - m) \psi_y y^\rho + \quad (35)$$

$$\int d^3x \left[\psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho \psi_y - \psi_x^\dagger \gamma^0 (i\partial_x - m) \psi_x x^\rho \psi_y \right] \quad (36)$$

$$= g^{0\lambda} \gamma^0 (i\partial_y - m) \psi_y y^\rho \quad (37)$$

However, the term $(i\cancel{\partial}_y - m)\psi(y)$ is just the Dirac equation and so is equal to zero. Thus

$$\left[\psi_y, \int d^3x g^{0\lambda} \mathcal{L} x^\rho \right] = 0 \quad (38)$$

Putting everything together gives us (and moving the indices to the bottom and changing the spacetime variable back from y to x so it matches L&P's equation 4.106)

$$[\psi, J_{\mu\nu}] = \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right) \psi \quad (39)$$

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