STRESS-ENERGY TENSOR AND ANGULAR MOMENTUM FOR DIRAC FIELD

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.26.

The stress-energy tensor appears in the definition of Noether's theorem and is defined in L&P's equation 2.44 as

$$T^{\mu\nu} \equiv \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)}\partial^\nu \Phi^A - g^{\mu\nu}\mathscr{L} \tag{1}$$

where there is a sum over the index A, which labels the independent fields Φ^A. $\mathscr{L}$ is the Lagrangian density and $g^{\mu\nu}$ is the metric tensor in flat spacetime. The stress-energy tensor is used to define a current

$$J^\mu \equiv \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)}\delta\Phi^A - T^{\mu\nu}\delta x_\nu \tag{2}$$

For the Dirac field, L&P use the Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{3}$$

If the action is invariant under some infinitesimal transformation which gives rise to variations $\delta\Phi^A$ in the fields and δx_ν in the coordinates, then the corresponding current is conserved, so that

$$\partial_\mu J^\mu = 0 \tag{4}$$

Starting with L&P's equation 2.56, there is a discussion of the conserved current if the Lagrangian is invariant under a Lorentz transformation. The variation in the field is given by their equation 2.57 as

$$\delta\Phi^A(x) = \frac{1}{2}\omega_{\lambda\rho}\left(\Sigma^{\lambda\rho}\right)^A_{\ B}\Phi^B \tag{5}$$

where $\omega_{\lambda\rho}$ are the components (independent of x) of the Lorentz transformation. The quantity $\Sigma^{\lambda\rho}$ is stated to be a 'spin matrix', although no other mention is made of it, nor are any examples given of what it is. I have to confess that I don't understand this, nor do I understand what the indexes A

and B mean with regard to $\Sigma^{\lambda\rho}$. In any case, they go on to show that as a result of Noether's theorem, we can state

$$\partial_\mu \mathscr{M}^{\mu\lambda\rho} = 0 \tag{6}$$

where

$$\mathscr{M}^{\mu\lambda\rho} = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)} \left(\Sigma^{\lambda\rho}\right)^A_B \Phi^B - \left(T^{\mu\lambda} x^\rho - T^{\mu\rho} x^\lambda\right) \tag{7}$$

and the corresponding conserved charge is

$$J^{\lambda\rho} = \int d^3x\, \mathscr{M}^{0\lambda\rho} \tag{8}$$

The problem in L&P's exercise 4.26 is to calculate $\mathscr{M}^{\mu\lambda\rho}$ and hence $J^{\lambda\rho}$ for the Dirac lagrangian 3. The stress-energy tensor is fairly straightforward. Since only derivatives with respect to ψ occur in 3, we have

$$T^{\mu\nu} = \overline{\psi} i\gamma^\mu \partial^\nu \psi - g^{\mu\nu} \mathscr{L} \tag{9}$$

To calculate $\mathscr{M}^{\mu\lambda\rho}$, however, we need to know what $\Sigma^{\lambda\rho}$ is. L&P state just before equation 4.72 that $\frac{1}{2}\sigma_{\mu\nu}$ is the spin operator for fermions, but if we use this as $\Sigma_{\mu\nu}$, we don't get the answer given in the back of the book. It appears that we need to use

$$\Sigma^{\lambda\rho} = -\frac{i}{2}\sigma^{\lambda\rho} \tag{10}$$

If anyone knows why this is so, or can give an explanation of exactly what $\Sigma^{\lambda\rho}$ is, I'd be most grateful.

In any case, using 10, we have from 7

$$\mathscr{M}^{\mu\lambda\rho} = \overline{\psi}\gamma^\mu \frac{1}{2}\sigma^{\lambda\rho}\psi - \left(T^{\mu\lambda} x^\rho - T^{\mu\rho} x^\lambda\right) \tag{11}$$

To calculate $J^{\lambda\rho}$ from 8, we need $T^{0\nu}$ which we get from 9:

$$T^{0\nu} = \overline{\psi} i\gamma^0 \partial^\nu \psi - g^{0\nu} \mathscr{L} \tag{12}$$

$$= i\psi^\dagger \partial^\nu \psi - g^{0\nu} \mathscr{L} \tag{13}$$

using the properties $\overline{\psi} = \psi^\dagger \gamma^0$ and $\left(\gamma^0\right)^2 = 1$. We therefore have

$$\mathscr{M}^{0\lambda\rho} = \frac{1}{2}\psi^\dagger \sigma^{\lambda\rho}\psi - i\psi^\dagger \partial^\lambda \psi x^\rho + g^{0\lambda} \mathscr{L} x^\rho + \tag{14}$$

$$i\psi^\dagger \partial^\rho \psi x^\lambda - g^{0\rho} \mathscr{L} x^\lambda \tag{15}$$

To calculate $\left[\psi,J_{\mu\nu}\right]$ we need the results

$$\left\{\psi\left(t,\mathbf{x}\right),\psi^{\dagger}\left(t,\mathbf{y}\right)\right\}=\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{16}$$
$$\left\{\psi\left(t,\mathbf{x}\right),\psi\left(t,\mathbf{y}\right)\right\}=0 \tag{17}$$

[Remember that this is actually a matrix equation, with components $\left\{\psi_{a}\left(t,\mathbf{x}\right),\psi_{b}^{\dagger}\left(t,\mathbf{y}\right)\right\}=\delta_{ab}\delta^{3}\left(\mathbf{x}-\mathbf{y}\right)$.]

We can apply 16 to the various terms in 14 when placed in the integral 8. We'll consider each term separately.

First, we look at

$$\left[\psi\left(y\right),\int d^{3}x\,\frac{1}{2}\psi^{\dagger}\left(x\right)\sigma^{\lambda\rho}\psi\left(x\right)\right] \tag{18}$$

To simplify the notation, I'll indicate dependence on x or y by a subscript rather than parentheses, so that $\psi_{y}\equiv\psi\left(y\right)$ and so on. We have

$$\left[\psi\left(y\right),\int d^{3}x\,\frac{1}{2}\psi^{\dagger}\left(x\right)\sigma^{\lambda\rho}\psi\left(x\right)\right]=\frac{1}{2}\int d^{3}x\,\left[\psi_{y}\psi_{x}^{\dagger}\sigma^{\lambda\rho}\psi_{x}-\psi_{x}^{\dagger}\sigma^{\lambda\rho}\psi_{x}\psi_{y}\right] \tag{19}$$
$$=\frac{1}{2}\int d^{3}x\left[\delta^{3}\left(\mathbf{x}-\mathbf{y}\right)\sigma^{\lambda\rho}\psi_{x}- \right. \tag{20}$$
$$\left.\psi_{x}^{\dagger}\psi_{y}\sigma^{\lambda\rho}\psi_{x}-\psi_{x}^{\dagger}\sigma^{\lambda\rho}\psi_{x}\psi_{y}\right] \tag{21}$$
$$=\frac{1}{2}\sigma^{\lambda\rho}\psi_{y}+\int d^{3}x\left[\psi_{x}^{\dagger}\sigma^{\lambda\rho}\psi_{x}\psi_{y}-\psi_{x}^{\dagger}\sigma^{\lambda\rho}\psi_{x}\psi_{y}\right] \tag{22}$$
$$=\frac{1}{2}\sigma^{\lambda\rho}\psi\left(y\right) \tag{23}$$

Next, we look at

$$\left[\psi_y, \int d^3x\, i\psi_x^\dagger \partial^\lambda \psi_x x^\rho\right] = i\int d^3x\, \left(\psi_y \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y\right) \tag{24}$$

$$= i\int d^3x\, \left(\delta^3(\mathbf{x}-\mathbf{y})\, \partial_x^\lambda \psi_x x^\rho - \right. \tag{25}$$

$$\left. \psi_x^\dagger \psi_y \partial_x^\lambda \psi_x x^\rho - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y\right) \tag{26}$$

$$= i\partial_y^\lambda \psi_y y^\rho + \int d^3x \left(\psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y - \psi_x^\dagger \partial_x^\lambda \psi_x x^\rho \psi_y\right) \tag{27}$$

$$= i\partial_y^\lambda \psi(y)\, y^\rho \tag{28}$$

Note that, in these equations, the derivatives are with respect to the integration variable x and thus do not affect ψ_y, so it is valid to anticommute ψ_y and $\partial^\lambda \psi_x$.

Finally we need to consider

$$\left[\psi_y, \int d^3x\, g^{0\lambda} \mathscr{L} x^\rho\right] = g^{0\lambda}\left[\psi_y, \int d^3x\, \overline{\psi}_x \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho\right] \tag{29}$$

$$= g^{0\lambda}\left[\psi_y, \int d^3x\, \psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho\right] \tag{30}$$

$$= g^{0\lambda} \int d^3x\, \left(\psi_y \psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho - \right. \tag{31}$$

$$\left. \psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho \psi_y\right) \tag{32}$$

$$= g^{0\lambda}\left\{\int d^3x\, \delta^3(\mathbf{x}-\mathbf{y})\, \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho - \right. \tag{33}$$

$$\left. \int d^3x \left[\psi_x^\dagger \gamma^0 \psi_y \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho - \psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho \psi_y\right]\right\} \tag{34}$$

$$= g^{0\lambda} \gamma^0 \left(i\partial\!\!\!/_y - m\right) \psi_y y^\rho + \tag{35}$$

$$\int d^3x \left[\psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho \psi_y - \psi_x^\dagger \gamma^0 \left(i\partial\!\!\!/_x - m\right) \psi_x x^\rho \psi_y\right] \tag{36}$$

$$= g^{0\lambda} \gamma^0 \left(i\partial\!\!\!/_y - m\right) \psi_y y^\rho \tag{37}$$

However, the term $\left(i\partial\!\!\!/_y - m\right)\psi(y)$ is just the Dirac equation and so is equal to zero. Thus

$$\left[\psi_y, \int d^3x\, g^{0\lambda}\mathscr{L}x^\rho\right] = 0 \tag{38}$$

Putting everything together gives us (and moving the indices to the bottom and changing the spacetime variable back from y to x so it matches L&P's equation 4.106)

$$\left[\psi, J_{\mu\nu}\right] = \left(i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + \frac{1}{2}\sigma_{\mu\nu}\right)\psi \tag{39}$$

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