ALTERNATIVE NORMALIZATION FOR DIRAC SPINORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.27.

The Dirac spinors are usually normalized so that

$$u_r^{\dagger}(\mathbf{p})\,u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p})\,v_s(\mathbf{p}) = 2E_p\delta_{rs} \tag{1}$$

$$v_r^{\dagger}(\mathbf{p})\,u_s(-\mathbf{p}) = u_r^{\dagger}(\mathbf{p})\,v_s(-\mathbf{p}) = 0 \tag{2}$$

The Dirac fields can be written as a Fourier expansion over the spinors, with creation and annihilation operators as coefficients

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s(\mathbf{p})\,u_s(\mathbf{p})\,e^{-ip\cdot x} + \hat{f}_s^{\dagger}(\mathbf{p})\,v_s(\mathbf{p})\,e^{ip\cdot x} \right) \tag{3}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s^{\dagger}(\mathbf{p})\,\overline{u}_s(\mathbf{p})\,e^{ip\cdot x} + \hat{f}_s(\mathbf{p})\,\overline{v}_s(\mathbf{p})\,e^{-ip\cdot x} \right) \tag{4}$$

The operators must obey the anticommutation relations

$$\left\{ f_r(\mathbf{p}), f_s^{\dagger}(\mathbf{q}) \right\} = \left\{ \hat{f}_r(\mathbf{p}), \hat{f}_s^{\dagger}(\mathbf{q}) \right\} = \delta_{rs}\delta^3(\mathbf{p}-\mathbf{q}) \tag{5}$$

with all other anticommutators being zero. Using these relations, we can show that the fields themselves obey the anticommutation relations

$$\left\{ \psi(t,\mathbf{x}), \psi^{\dagger}(t,\mathbf{y}) \right\} = \delta^3(\mathbf{x}-\mathbf{y}) \tag{6}$$

There are other conventions for the normalization constants in these relations. We can replace 1 by

$$u_r^{\dagger}(\mathbf{p})\,u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p})\,v_s(\mathbf{p}) = N_1\delta_{rs} \tag{7}$$

and 3 and 4 by

$$\psi(x) = \int \frac{d^3p}{\sqrt{N_3(2\pi)^3}} \sum_{s=1,2} \left(f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip\cdot x} \right) \quad (8)$$

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{N_3(2\pi)^3}} \sum_{s=1,2} \left(f_s^\dagger(\mathbf{p}) \overline{u}_s(\mathbf{p}) e^{ip\cdot x} + \hat{f}_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{-ip\cdot x} \right) \quad (9)$$

and finally replace 5 by

$$\left\{ f_r(\mathbf{p}), f_s^\dagger(\mathbf{q}) \right\} = \left\{ \hat{f}_r(\mathbf{p}), \hat{f}_s^\dagger(\mathbf{q}) \right\} = N_2 \delta_{rs} \delta^3(\mathbf{p}-\mathbf{q}) \quad (10)$$

If we still want 6 to be true, this imposes a restriction on the constants N_1, N_2 and N_3. To see this, we need to retrace the derivation of 6. To begin, first look at the derivation following equation (20) in the earlier post. We had

$$\overline{u}_s(\mathbf{p}) p^\mu u_r(\mathbf{p}) = m\overline{u}_s(\mathbf{p}) \gamma^\mu u_r(\mathbf{p}) \quad (11)$$

The $\mu = 0$ component of the RHS of this equation leads to, using the new normalization 7:

$$\begin{aligned} m\overline{u}_s(\mathbf{p})\gamma^0 u_r(\mathbf{p}) &= m u_s^\dagger(\mathbf{p}) \left(\gamma^0\right)^2 u_r(\mathbf{p}) &(12)\\ &= m u_s^\dagger(\mathbf{p}) u_r(\mathbf{p}) &(13)\\ &= m N_1 \delta_{rs} &(14)\end{aligned}$$

Setting this equal to the LHS of 11 gives us

$$\begin{aligned} mN_1\delta_{rs} &= \overline{u}_s(\mathbf{p}) p^0 u_r(\mathbf{p}) &(15)\\ &= E_p \overline{u}_s(\mathbf{p}) u_r(\mathbf{p}) &(16)\end{aligned}$$

so we get

$$\overline{u}_s(\mathbf{p}) u_r(\mathbf{p}) = \frac{mN_1}{E_p}\delta_{rs} \quad (17)$$

In the original derivation of 6, we made use of the identities

$$\begin{aligned} \sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= \not{p} + m &(18)\\ \sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= \not{p} - m &(19)\end{aligned}$$

If we follow through the derivation of these identities using the new normalization, we have

$$\left[\sum_s u_s \overline{u}_s\right] u_- = u_+ \overline{u}_+ u_- + u_- \overline{u}_- u_- \tag{20}$$

$$= 0 + \frac{mN_1}{E_p} u_- \tag{21}$$

$$= \frac{mN_1}{E_p} u_- \tag{22}$$

However, the RHS of 18 still gives us

$$\left(\not{p} + m\right) u_- = \left(\not{p} - m + 2m\right) u_- \tag{23}$$

$$= \left(\not{p} - m\right) u_- + 2mu_- \tag{24}$$

$$= 2mu_- \tag{25}$$

so the modified form of 18 is

$$\sum_s u_s(\mathbf{p}) \overline{u}_s(\mathbf{p}) = \frac{N_1}{2E_p} \left(\not{p} + m\right) \tag{26}$$

The same extra factor appears in the equations for $v_s(\mathbf{p})$.

When this modification is inserted into the derivation of 6, an extra factor of $\frac{N_1}{2E_p}$ is inserted. Because of 10, there is also an extra factor of N_2 and because of 8 and 9 the factor of $2E_p$ in the denominator of the original anticommutator is replaced by N_3 (since there is a product of a ψ and $\psi^\dagger$ in the anticommutator, so we have a factor of $\left(\sqrt{N_3}\right)^2$ in the denominator). Thus the net effect is the replacement

$$\frac{1}{2E_p} \to \frac{N_1 N_2}{2E_p N_3} \tag{27}$$

If we want 6 to remain unchanged, we therefore require

$$N_1 N_2 = N_3 \tag{28}$$