

## ALTERNATIVE NORMALIZATION FOR DIRAC SPINORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.27.

The Dirac spinors are usually normalized so that

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = 2E_p \delta_{rs} \quad (1)$$

$$v_r^\dagger(\mathbf{p}) u_s(-\mathbf{p}) = u_r^\dagger(\mathbf{p}) v_s(-\mathbf{p}) = 0 \quad (2)$$

The Dirac fields can be written as a Fourier expansion over the spinors, with creation and annihilation operators as coefficients

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left( f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip \cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip \cdot x} \right) \quad (3)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left( f_s^\dagger(\mathbf{p}) \bar{u}_s(\mathbf{p}) e^{ip \cdot x} + \hat{f}_s(\mathbf{p}) \bar{v}_s(\mathbf{p}) e^{-ip \cdot x} \right) \quad (4)$$

The operators must obey the anticommutation relations

$$\left\{ f_r(\mathbf{p}), f_s^\dagger(\mathbf{q}) \right\} = \left\{ \hat{f}_r(\mathbf{p}), \hat{f}_s^\dagger(\mathbf{q}) \right\} = \delta_{rs} \delta^3(\mathbf{p} - \mathbf{q}) \quad (5)$$

with all other anticommutators being zero. Using these relations, we can show that the fields themselves obey the anticommutation relations

$$\left\{ \psi(t, \mathbf{x}), \psi^\dagger(t, \mathbf{y}) \right\} = \delta^3(\mathbf{x} - \mathbf{y}) \quad (6)$$

There are other conventions for the normalization constants in these relations. We can replace 1 by

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = N_1 \delta_{rs} \quad (7)$$

and 3 and 4 by

$$\psi(x) = \int \frac{d^3p}{\sqrt{N_3(2\pi)^3}} \sum_{s=1,2} \left( f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip \cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip \cdot x} \right) \quad (8)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{\sqrt{N_3(2\pi)^3}} \sum_{s=1,2} \left( f_s^\dagger(\mathbf{p}) \bar{u}_s(\mathbf{p}) e^{ip \cdot x} + \hat{f}_s(\mathbf{p}) \bar{v}_s(\mathbf{p}) e^{-ip \cdot x} \right) \quad (9)$$

and finally replace 5 by

$$\left\{ f_r(\mathbf{p}), f_s^\dagger(\mathbf{q}) \right\} = \left\{ \hat{f}_r(\mathbf{p}), \hat{f}_s^\dagger(\mathbf{q}) \right\} = N_2 \delta_{rs} \delta^3(\mathbf{p} - \mathbf{q}) \quad (10)$$

If we still want 6 to be true, this imposes a restriction on the constants  $N_1$ ,  $N_2$  and  $N_3$ . To see this, we need to retrace the derivation of 6. To begin, first look at the derivation following equation (20) in the earlier post. We had

$$\bar{u}_s(\mathbf{p}) p^\mu u_r(\mathbf{p}) = m \bar{u}_s(\mathbf{p}) \gamma^\mu u_r(\mathbf{p}) \quad (11)$$

The  $\mu = 0$  component of the RHS of this equation leads to, using the new normalization 7:

$$m \bar{u}_s(\mathbf{p}) \gamma^0 u_r(\mathbf{p}) = m u_s^\dagger(\mathbf{p}) (\gamma^0)^2 u_r(\mathbf{p}) \quad (12)$$

$$= m u_s^\dagger(\mathbf{p}) u_r(\mathbf{p}) \quad (13)$$

$$= m N_1 \delta_{rs} \quad (14)$$

Setting this equal to the LHS of 11 gives us

$$m N_1 \delta_{rs} = \bar{u}_s(\mathbf{p}) p^0 u_r(\mathbf{p}) \quad (15)$$

$$= E_p \bar{u}_s(\mathbf{p}) u_r(\mathbf{p}) \quad (16)$$

so we get

$$\bar{u}_s(\mathbf{p}) u_r(\mathbf{p}) = \frac{m N_1}{E_p} \delta_{rs} \quad (17)$$

In the original derivation of 6, we made use of the identities

$$\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) = \not{p} + m \quad (18)$$

$$\sum_s v_s(\mathbf{p}) \bar{v}_s(\mathbf{p}) = \not{p} - m \quad (19)$$

If we follow through the derivation of these identities using the new normalization, we have

$$\left[ \sum_s u_s \bar{u}_s \right] u_- = u_+ \bar{u}_+ u_- + u_- \bar{u}_- u_- \quad (20)$$

$$= 0 + \frac{mN_1}{E_p} u_- \quad (21)$$

$$= \frac{mN_1}{E_p} u_- \quad (22)$$

However, the RHS of 18 still gives us

$$(\not{p} + m) u_- = (\not{p} - m + 2m) u_- \quad (23)$$

$$= (\not{p} - m) u_- + 2m u_- \quad (24)$$

$$= 2m u_- \quad (25)$$

so the modified form of 18 is

$$\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) = \frac{N_1}{2E_p} (\not{p} + m) \quad (26)$$

The same extra factor appears in the equations for  $v_s(\mathbf{p})$ .

When this modification is inserted into the derivation of 6, an extra factor of  $\frac{N_1}{2E_p}$  is inserted. Because of 10, there is also an extra factor of  $N_2$  and because of 8 and 9 the factor of  $2E_p$  in the denominator of the original anticommutator is replaced by  $N_3$  (since there is a product of a  $\psi$  and  $\psi^\dagger$  in the anticommutator, so we have a factor of  $(\sqrt{N_3})^2$  in the denominator). Thus the net effect is the replacement

$$\frac{1}{2E_p} \rightarrow \frac{N_1 N_2}{2E_p N_3} \quad (27)$$

If we want 6 to remain unchanged, we therefore require

$$N_1 N_2 = N_3 \quad (28)$$