HELICITY OF ONE-PARTICLE DIRAC STATES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.28.

We can invert the Fourier expansions of the Dirac fields to get expressions for the creation and annihilation operators in terms of the fields:

There is a typo in L&P's first eqn 4.112, where the exponent should be $+ik\cdot x$.

$$f_r(\mathbf{k})=\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{ik\cdot x}u_r^{\dagger}(\mathbf{k})\psi(x) \tag{1}$$

$$\hat{f}_r^{\dagger}(\mathbf{k})=\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{-ik\cdot x}v_r^{\dagger}(\mathbf{k})\psi(x) \tag{2}$$

To examine the helicity (the component of spin parallel to the momentum), we look at the operator

$$\mathbf{J_k}=\frac{\mathbf{J}\cdot\mathbf{k}}{|\mathbf{k}|} \tag{3}$$

where $\mathbf{J}$ is the angular momentum operator. For a free particle, there is no orbital angular momentum, so $\mathbf{J}$ consists entirely of spin. We can use the formula

$$[\psi, J_{\mu\nu}]=\left(i(x_\mu\partial_\nu-x_\nu\partial_\mu)+\frac{1}{2}\sigma_{\mu\nu}\right)\psi \tag{4}$$

to derive the commutator

$$[\psi(x),\mathbf{J_k}]=\frac{1}{2}\Sigma_{\mathbf{k}}\psi(x) \tag{5}$$

where

$$\Sigma_{\mathbf{k}}\equiv\frac{\Sigma\cdot\mathbf{k}}{|\mathbf{k}|} \tag{6}$$

and Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma\equiv\left(\sigma^{23},\sigma^{31},\sigma^{12}\right) \tag{7}$$

L&P show in their eqns 4.115 and 4.116 that

$$\left[\mathbf{J_k},\hat{f}_r^\dagger(\mathbf{k})\right]=\frac{1}{2}r\hat{f}_r^\dagger(\mathbf{k}) \tag{8}$$

Since $\mathbf{J_k}|0\rangle=0$ as the vacuum state has no angular momentum of any sort, we have

$$\begin{aligned}\left[\mathbf{J_k},\hat{f}_r^\dagger(\mathbf{k})\right]|0\rangle &= \left(\mathbf{J_k}\hat{f}_r^\dagger(\mathbf{k})-\hat{f}_r^\dagger(\mathbf{k})\mathbf{J_k}\right)|0\rangle && (9)\\ &=\mathbf{J_k}\hat{f}_r^\dagger(\mathbf{k})|0\rangle && (10)\\ &=\frac{1}{2}r\hat{f}_r^\dagger(\mathbf{k})|0\rangle && (11)\end{aligned}$$

Thus the one-particle state $\hat{f}_r^\dagger(\mathbf{k})|0\rangle$ has an eigenvalue of $\frac{r}{2}$ with respect to the operator $\mathbf{J}_k$. In other words, the helicity of the state $\hat{f}_r^\dagger(\mathbf{k})|0\rangle$ is r, where $r=\pm1$.

We can show the same thing for the state $f_r^\dagger(\mathbf{k})$. From 1, we have

$$f_r^\dagger(\mathbf{k})=\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{ik\cdot x}\psi^\dagger(x)u_r(\mathbf{k}) \tag{12}$$

Since the angular momentum is hermitian, we have

$$\begin{aligned}\left[\psi(x),\mathbf{J_k}\right]^\dagger &= \left(\psi(x)\mathbf{J_k}-\mathbf{J_k}\psi(x)\right)^\dagger && (13)\\ &=\mathbf{J_k}\psi^\dagger(x)-\psi^\dagger(x)\mathbf{J_k} && (14)\\ &=\left[\mathbf{J_k},\psi^\dagger(x)\right] && (15)\end{aligned}$$

So, from 5 we have, since helicity is also hermitian:

$$\left[\mathbf{J_k},\psi^\dagger(x)\right]=\frac{1}{2}\psi^\dagger(x)\Sigma_\mathbf{k} \tag{16}$$

Therefore, we have

$$\begin{aligned}\left[f_r^\dagger(\mathbf{k}),\mathbf{J_k}\right] &= \frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{ik\cdot x}\left[\psi^\dagger(x),\mathbf{J_k}\right]u_r(\mathbf{k}) && (17)\\ &=-\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{ik\cdot x}\left[\mathbf{J_k},\psi^\dagger(x)\right]u_r(\mathbf{k}) && (18)\\ &=-\frac{1}{2}\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{ik\cdot x}\psi^\dagger(x)\Sigma_\mathbf{k}u_r(\mathbf{k}) && (19)\end{aligned}$$

From L&P's eqn 4.77, the spinors u_r are eigenstates of $\Sigma_{\mathbf{k}}$ with eigenvalue r, so we have, using 12

$$\left[f_r^\dagger(\mathbf{k}),\mathbf{J}_{\mathbf{k}}\right] = -\frac{r}{2}\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\; e^{ik\cdot x}\psi^\dagger(x)\,u_r(\mathbf{k}) \tag{20}$$

$$= -\frac{r}{2}f_r^\dagger(\mathbf{k}) \tag{21}$$

or

$$\left[\mathbf{J}_{\mathbf{k}}, f_r^\dagger(\mathbf{k})\right] = \frac{r}{2}f_r^\dagger(\mathbf{k}) \tag{22}$$

By the same calculation as in 11, we see that the state $f_r^\dagger(\mathbf{k})\left|0\right\rangle$ is a state of helicity r.