

HELICITY OF ONE-PARTICLE DIRAC STATES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.28.

We can invert the Fourier expansions of the Dirac fields to get expressions for the creation and annihilation operators in terms of the fields:

$$f_r(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} u_r^\dagger(\mathbf{k}) \psi(x) \quad (1)$$

$$\hat{f}_r^\dagger(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{-ik \cdot x} v_r^\dagger(\mathbf{k}) \psi(x) \quad (2)$$

There is a typo in L&P's first eqn 4.112, where the exponent should be $+ik \cdot x$.

To examine the helicity (the component of spin parallel to the momentum), we look at the operator

$$\mathbf{J}_k = \frac{\mathbf{J} \cdot \mathbf{k}}{|\mathbf{k}|} \quad (3)$$

where \mathbf{J} is the angular momentum operator. For a free particle, there is no orbital angular momentum, so \mathbf{J} consists entirely of spin. We can use the formula

$$[\psi, J_{\mu\nu}] = \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right) \psi \quad (4)$$

to derive the commutator

$$[\psi(x), \mathbf{J}_k] = \frac{1}{2} \Sigma_k \psi(x) \quad (5)$$

where

$$\Sigma_k \equiv \frac{\Sigma \cdot \mathbf{k}}{|\mathbf{k}|} \quad (6)$$

and Σ is a 3-vector defined in terms of the spin matrices as

$$\Sigma \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12}) \quad (7)$$

L&P show in their eqns 4.115 and 4.116 that

$$\left[\mathbf{J}_{\mathbf{k}}, \hat{f}_r^\dagger(\mathbf{k}) \right] = \frac{1}{2} r \hat{f}_r^\dagger(\mathbf{k}) \quad (8)$$

Since $\mathbf{J}_{\mathbf{k}}|0\rangle = 0$ as the vacuum state has no angular momentum of any sort, we have

$$\left[\mathbf{J}_{\mathbf{k}}, \hat{f}_r^\dagger(\mathbf{k}) \right] |0\rangle = \left(\mathbf{J}_{\mathbf{k}} \hat{f}_r^\dagger(\mathbf{k}) - \hat{f}_r^\dagger(\mathbf{k}) \mathbf{J}_{\mathbf{k}} \right) |0\rangle \quad (9)$$

$$= \mathbf{J}_{\mathbf{k}} \hat{f}_r^\dagger(\mathbf{k}) |0\rangle \quad (10)$$

$$= \frac{1}{2} r \hat{f}_r^\dagger(\mathbf{k}) |0\rangle \quad (11)$$

Thus the one-particle state $\hat{f}_r^\dagger(\mathbf{k})|0\rangle$ has an eigenvalue of $\frac{r}{2}$ with respect to the operator $\mathbf{J}_{\mathbf{k}}$. In other words, the helicity of the state $\hat{f}_r^\dagger(\mathbf{k})|0\rangle$ is r , where $r = \pm 1$.

We can show the same thing for the state $f_r^\dagger(\mathbf{k})$. From 1, we have

$$f_r^\dagger(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} \psi^\dagger(x) u_r(\mathbf{k}) \quad (12)$$

Since the angular momentum is hermitian, we have

$$[\psi(x), \mathbf{J}_{\mathbf{k}}]^\dagger = (\psi(x) \mathbf{J}_{\mathbf{k}} - \mathbf{J}_{\mathbf{k}} \psi(x))^\dagger \quad (13)$$

$$= \mathbf{J}_{\mathbf{k}} \psi^\dagger(x) - \psi^\dagger(x) \mathbf{J}_{\mathbf{k}} \quad (14)$$

$$= \left[\mathbf{J}_{\mathbf{k}}, \psi^\dagger(x) \right] \quad (15)$$

So, from 5 we have, since helicity is also hermitian:

$$\left[\mathbf{J}_{\mathbf{k}}, \psi^\dagger(x) \right] = \frac{1}{2} \psi^\dagger(x) \Sigma_{\mathbf{k}} \quad (16)$$

Therefore, we have

$$\left[f_r^\dagger(\mathbf{k}), \mathbf{J}_{\mathbf{k}} \right] = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} \left[\psi^\dagger(x), \mathbf{J}_{\mathbf{k}} \right] u_r(\mathbf{k}) \quad (17)$$

$$= -\frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} \left[\mathbf{J}_{\mathbf{k}}, \psi^\dagger(x) \right] u_r(\mathbf{k}) \quad (18)$$

$$= -\frac{1}{2} \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} \psi^\dagger(x) \Sigma_{\mathbf{k}} u_r(\mathbf{k}) \quad (19)$$

From L&P's eqn 4.77, the spinors u_r are eigenstates of $\Sigma_{\mathbf{k}}$ with eigenvalue r , so we have, using 12

$$\left[f_r^\dagger(\mathbf{k}), \mathbf{J}_{\mathbf{k}} \right] = -\frac{r}{2} \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x e^{ik \cdot x} \psi^\dagger(x) u_r(\mathbf{k}) \quad (20)$$

$$= -\frac{r}{2} f_r^\dagger(\mathbf{k}) \quad (21)$$

or

$$\left[\mathbf{J}_{\mathbf{k}}, f_r^\dagger(\mathbf{k}) \right] = \frac{r}{2} f_r^\dagger(\mathbf{k}) \quad (22)$$

By the same calculation as in 11, we see that the state $f_r^\dagger(\mathbf{k})|0\rangle$ is a state of helicity r .