PROPAGATOR FOR THE DIRAC FIELD AS A TIME-ORDERED PRODUCT

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 4, Problem 4.30.

We can use a similar method to that for the time-ordering of terms in the propagator for the scalar field to derive an expression for the propagator for the Dirac field as a time-ordered product. The propagator for the Dirac field is

$$iS_F\left(x-x'\right)=\int\frac{d^3p}{\left(2\pi\right)^3 2E_p}\left[\Theta\left(t-t'\right)\left(\not{p}+m\right)e^{-ip\cdot\left(x-x'\right)}-\right. \tag{1}$$

$$\left.\Theta\left(t'-t\right)\left(\not{p}-m\right)e^{ip\cdot\left(x-x'\right)}\right] \tag{2}$$

Now consider the effects of applying the field operators to the vacuum state. The field operators are

$$\psi\left(x\right)=\int\frac{d^3p}{\sqrt{2\left(2\pi\right)^3 E_p}}\sum_{s=1,2}\left(f_s\left(\mathbf{p}\right)u_s\left(\mathbf{p}\right)e^{-ip\cdot x}+\hat{f}_s^{\dagger}\left(\mathbf{p}\right)v_s\left(\mathbf{p}\right)e^{ip\cdot x}\right) \tag{3}$$

$$\overline{\psi}\left(x\right)=\int\frac{d^3p}{\sqrt{2\left(2\pi\right)^3 E_p}}\sum_{s=1,2}\left(f_s^{\dagger}\left(\mathbf{p}\right)\overline{u}_s\left(\mathbf{p}\right)e^{ip\cdot x}+\hat{f}_s\left(\mathbf{p}\right)\overline{v}_s\left(\mathbf{p}\right)e^{-ip\cdot x}\right) \tag{4}$$

When applying these operators to the vacuum ket, all terms containing annihilation operators will give zero, so we need look only at terms containing creation operators. We'll also look at individual components of the fields (remember that each ψ is a 4-component vector). We have

$$\overline{\psi}_\beta\left(x'\right)\left|0\right\rangle=\int\frac{d^3p}{\sqrt{2\left(2\pi\right)^3 E_p}}\sum_{r=1,2}f_r^{\dagger}\left(\mathbf{p}\right)\overline{u}_{r\beta}\left(\mathbf{p}\right)e^{ip\cdot x'}\left|0\right\rangle \tag{5}$$

Applying the fields to the vacuum bra, only annihilation operators give non-zero results, so we have

$$\langle 0|\psi_\alpha(x) = \langle 0|\int\frac{d^3p'}{\sqrt{2(2\pi)^3 E_{p'}}}\sum_{s=1,2} f_s(\mathbf{p}')\,u_{s\alpha}(\mathbf{p}')\,e^{-ip'\cdot x} \tag{6}$$

We now form the overall bracket:

$$\left\langle 0\left|\psi_\alpha(x')\,\overline{\psi}_\beta(x)\right|0\right\rangle = \langle 0|\int\frac{d^3p'}{\sqrt{2(2\pi)^3 E_{p'}}}\sum_{s=1,2} f_s(\mathbf{p}')\,u_{s\alpha}(\mathbf{p}')\times \tag{7}$$

$$\int\frac{d^3p}{\sqrt{2(2\pi)^3 E_p}}\sum_{r=1,2} f_r^\dagger(\mathbf{p})\,\overline{u}_{r\beta}(\mathbf{p})\,e^{-i(p'\cdot x-p\cdot x')}\,|0\rangle \tag{8}$$

We can now use the anticommutators

$$\left\{f_r(\mathbf{p}),f_s^\dagger(\mathbf{p}')\right\} = \delta_{rs}\delta^3(\mathbf{p}-\mathbf{p}') \tag{9}$$

This gives

$$f_s(\mathbf{p}')\,f_r^\dagger(\mathbf{p}) = \delta_{rs}\delta^3(\mathbf{p}-\mathbf{p}') - f_r^\dagger(\mathbf{p})\,f_s(\mathbf{p}') \tag{10}$$

When this is substituted into 8, the second term on the RHS gives zero because we now have an annihilation operator $f_s(\mathbf{p}')$ operating on a vacuum ket (and also a creation operator $f_r^\dagger(\mathbf{p})$ operating on a vacuum bra). The first term on the RHS allows us to do one of the sums and one of the integrals. We therefore have

$$\left\langle 0\left|\psi_\alpha(x)\,\overline{\psi}_\beta(x')\right|0\right\rangle = \left\langle 0\left|\int\frac{d^3p}{\sqrt{2(2\pi)^3 E_p}}\sum_s u_{s\alpha}(\mathbf{p})\,\overline{u}_{s\beta}(\mathbf{p})\,e^{-ip\cdot(x-x')}\right|0\right\rangle \tag{11}$$

We can now use the identity

Remember this is a matrix equation.

$$\sum_s u_s(\mathbf{p})\,\overline{u}_s(\mathbf{p}) = \not{p}+m \tag{12}$$

so we get

$$\left\langle 0\left|\psi_{\alpha}\left(x\right)\overline{\psi}_{\beta}\left(x^{\prime}\right)\right|0\right\rangle =\left\langle 0\left|\int\frac{d^{3}p}{\sqrt{2\left(2\pi\right)^{3}E_{p}}}\left(\not{p}+m\right)_{\alpha\beta}e^{-ip\cdot\left(x-x^{\prime}\right)}\right|0\right\rangle \tag{13}$$

This is the same as the first term in 1. We can do a similar calculation with the fields reversed in order. We get

$$\psi_{\alpha}\left(x\right)\left|0\right\rangle =\int\frac{d^{3}p}{\sqrt{2\left(2\pi\right)^{3}E_{p}}}\sum_{r=1,2}\hat{f}_{r}^{\dagger}\left(\mathbf{p}\right)v_{r\alpha}\left(\mathbf{p}\right)e^{ip\cdot x}\left|0\right\rangle \tag{14}$$

$$\left\langle 0\right|\overline{\psi}_{\beta}\left(x^{\prime}\right)=\left\langle 0\right|\int\frac{d^{3}p^{\prime}}{\sqrt{2\left(2\pi\right)^{3}E_{p^{\prime}}}}\sum_{s=1,2}\hat{f}_{s}\left(\mathbf{p}^{\prime}\right)\overline{v}_{s\beta}\left(\mathbf{p}^{\prime}\right)e^{-ip^{\prime}\cdot x^{\prime}} \tag{15}$$

Forming the overall bracket

$$\left\langle 0\left|\overline{\psi}_{\beta}\left(x^{\prime}\right)\psi_{\alpha}\left(x\right)\right|0\right\rangle =\left\langle 0\right|\int\frac{d^{3}p^{\prime}}{\sqrt{2\left(2\pi\right)^{3}E_{p^{\prime}}}}\sum_{s=1,2}\hat{f}_{s}\left(\mathbf{p}^{\prime}\right)\overline{v}_{s\beta}\left(\mathbf{p}^{\prime}\right)\times \tag{16}$$

$$\int\frac{d^{3}p}{\sqrt{2\left(2\pi\right)^{3}E_{p}}}\sum_{r=1,2}\hat{f}_{r}^{\dagger}\left(\mathbf{p}\right)v_{r\alpha}\left(\mathbf{p}\right)e^{-i\left(p^{\prime}\cdot x^{\prime}-p\cdot x\right)}\left|0\right\rangle \tag{17}$$

We can again use the anticommutator to get

$$\hat{f}_{s}\left(\mathbf{p}^{\prime}\right)\hat{f}_{r}^{\dagger}\left(\mathbf{p}\right)=\delta_{rs}\delta^{3}\left(\mathbf{p}-\mathbf{p}^{\prime}\right)-\hat{f}_{r}^{\dagger}\left(\mathbf{p}\right)\hat{f}_{s}\left(\mathbf{p}^{\prime}\right) \tag{18}$$

We also have the identity

$$\sum_{s}v_{s}\left(\mathbf{p}\right)\overline{v}_{s}\left(\mathbf{p}\right)=\not{p}-m \tag{19}$$

so we get

$$\left\langle 0\left|\overline{\psi}_{\beta}\left(x^{\prime}\right)\psi_{\alpha}\left(x\right)\right|0\right\rangle =\left\langle 0\left|\int\frac{d^{3}p}{\sqrt{2\left(2\pi\right)^{3}E_{p}}}\left(\not{p}-m\right)_{\alpha\beta}e^{ip\cdot\left(x-x^{\prime}\right)}\right|0\right\rangle \tag{20}$$

This matches the second term in 1, except for the minus sign. If we define the time ordered product for fermions as

$$\mathscr{T}\left[\psi_{\alpha}(x)\overline{\psi}_{\beta}\left(x^{\prime}\right)\right]\equiv\begin{cases}\psi_{\alpha}(x)\overline{\psi}_{\beta}(x^{\prime}) & \text{if } t>t^{\prime}\\ -\overline{\psi}_{\beta}(x^{\prime})\psi_{\alpha}(x) & \text{if } t^{\prime}>t\end{cases} \tag{21}$$

then we can write the propagator (in terms of its matrix elements α and β) as

$$iS_{F\alpha\beta}\left(x-x^{\prime}\right)=\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}(x)\overline{\psi}_{\beta}\left(x^{\prime}\right)\right]\right|0\right\rangle \tag{22}$$

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