

## EVOLUTION OPERATOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Section 5.2.

W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 8.

When dealing with interacting fields, the Lagrangian consists of two parts: that for the free fields (which describes the system before and after the interaction takes place), and an interaction term, which describes the interaction between the fields (particles). We have

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \quad (1)$$

We can derive the Hamiltonian from the Lagrangian according to the equation

$$\mathcal{H} = \pi(x) \dot{\phi}(x) - \mathcal{L}(x) \quad (2)$$

where  $\phi$  is the field operator and  $\pi$  is the conjugate momentum.

In practice, the interaction Hamiltonian is usually one for which the evolution of the system cannot be solved exactly, so some form of approximation must be used. In L&P's section 5.2, a clear discussion is given of how this is done, so we'll summarize the results and comment on a few points.

The starting point is to split the Hamiltonian into a solvable part  $H_0$  (which is taken to be for free fields in L&P, although they point out that there are some other cases where this portion of the Hamiltonian can be solved) and an interaction part  $H_I$ , so we have

$$H = H_0 + H_I \quad (3)$$

It is then stated that a quantum state  $|\Psi(t)\rangle$  evolves in time according to

$$i \frac{d}{dt} |\Psi(t)\rangle = (H_0 + H_I) |\Psi(t)\rangle \quad (4)$$

The thing that puzzles me about this is that this is essentially the Schrödinger equation which (I think) applies only to nonrelativistic quantum mechanics. It's not clear that a derivation based on the Schrödinger equation is valid for

relativistic equations such as the Klein-Gordon and Dirac equations. However, Greiner & Reinhardt also start with this equation without any further comment, so it seems it is the accepted method.

If there is no interaction, then the state is a superposition of states chosen from a basis of free states. We call this state  $|\Psi_0(t)\rangle$ . This state obeys the equation

$$i\frac{d}{dt}|\Psi_0(t)\rangle = H_0|\Psi_0(t)\rangle \quad (5)$$

We then define two separate evolution operators by the equations

$$|\Psi_0(t)\rangle = U_0(t)|\Psi_0(-\infty)\rangle \equiv U_0(t)|i\rangle \quad (6)$$

$$|\Psi(t)\rangle = U_0(t)U(t)U_0^\dagger(t)|\Psi_0(t)\rangle \quad (7)$$

where  $|\Psi_0(-\infty)\rangle \equiv |i\rangle$  is the state at  $t = -\infty$ , which is taken to be any time sufficiently previous to the interaction that no interaction is felt.

If the states are all normalized, then we must have

$$\langle\Psi_0(t)|\Psi_0(t)\rangle = \langle i|U_0^\dagger(t)U_0(t)|i\rangle = 1 \quad (8)$$

so that

$$U_0^\dagger(t)U_0(t) = 1 \quad (9)$$

$$U_0^\dagger(t) = U_0^{-1}(t) \quad (10)$$

and  $U_0$  is unitary. Similarly, we must have

$$\langle\Psi(t)|\Psi(t)\rangle = \langle\Psi_0(t)|U_0(t)U^\dagger(t)U_0^\dagger(t)U_0(t)U(t)U_0^\dagger(t)|\Psi_0(t)\rangle \quad (11)$$

$$= \langle\Psi_0(t)|U_0(t)U^\dagger(t)U(t)U_0^\dagger(t)|\Psi_0(t)\rangle \quad (12)$$

$$= \langle\Psi_0(-\infty)|U^\dagger(t)U(t)|\Psi_0(-\infty)\rangle \quad (13)$$

$$= 1 \quad (14)$$

which will be true if  $U$  is also unitary.

From here, L&P give a detailed derivation from their equations 5.12 to 5.23, resulting in a differential equation for  $U(t)$ :

$$i\frac{dU(t)}{dt} = U_0^\dagger(t)H_I U_0(t)U(t) \quad (15)$$

$$= H_I(t)U(t) \quad (16)$$

where

$$H_I(t) \equiv U_0^\dagger(t)H_I U_0(t) \quad (17)$$

The operator  $H_I(t)$  is actually the interaction Hamiltonian in the *interaction picture*, as described by Greiner & Reinhardt in section 8.2. L&P make only a passing mention of this just before eqn 5.27, so if you want a more complete discussion, have a look at Greiner & Reinhardt.

In a practical problem (that is, a calculation that can be compared with experiment), we're assumed to know  $H_I(t)$  and we want to find  $U(t)$ . The problem with 16 is that  $H_I(t)$  depends on time, so we can't just write down a solution of the form

$$U(t) = U(0)e^{-iH_I t} \quad (18)$$

The approach taken is to derive an iterative formula based on successive approximations. We begin with the assumption that the basis states  $|\beta\rangle$  of the free Hamiltonian  $H_0$  are also a complete set of eigenstates of the full Hamiltonian  $H$ , which amounts to assuming that as  $t \rightarrow -\infty$ ,  $H_I \rightarrow 0$  and thus  $U(t) \rightarrow 1$ . This seems reasonable, since for times long before the interaction, the particles will be far enough apart that they are essentially free.

We can write a formal 'solution' to 16 in the form of an integral equation:

$$U(t) = U(t_0) - i \int_{t_0}^t dt_1 H_I(t_1)U(t_1) \quad (19)$$

Note that the variable  $t$  appears on the RHS only as the upper limit of the integral, so if we take the derivative, we get

$$\frac{dU(t)}{dt} = -i \frac{d}{dt} \left[ \int_{t_0}^t dt_1 H_I(t_1)U(t_1) \right] \quad (20)$$

$$= -iH_I(t)U(t) \quad (21)$$

which agrees with 16. If we extend  $t_0 \rightarrow -\infty$  so that  $U(t_0) \rightarrow 1$ , we have

$$U(t) = 1 - i \int_{-\infty}^t dt_1 H_I(t_1)U(t_1) \quad (22)$$

This 'solution' isn't much use on its own, since the operator  $U(t)$  appears on both sides of the equation, and, what's worse, as part of the integrand on

the RHS. However, we can repeat the process iteratively to obtain successively better solutions. The next step would involve replacing  $U(t_1)$  inside the integral in 22 by the entire RHS of 22 (and using another dummy integration variable  $t_2$  in this replacement):

$$U(t) = 1 - i \int_{-\infty}^t dt_1 H_I(t_1) \left( 1 - i \int_{-\infty}^{t_1} dt_2 H_I(t_2) U(t_2) \right) \quad (23)$$

$$= 1 + (-i) \int_{-\infty}^t dt_1 H_I(t_1) + \quad (24)$$

$$(-i)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) U(t_2) \quad (25)$$

Notice a couple of things about this second order expression. The first two terms no longer contain  $U(t)$  at all, so in principle can be evaluated. In the last term, with the double integral, the integral over  $t_2$  is only up to  $t_1$ , while the integral over  $t_1$  is up to the required final time  $t$ .

We can continue the iteration to infinity and get

$$U(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \quad (26)$$

$$H_I(t_1) H_I(t_2) \dots H_I(t_n) \quad (27)$$

This multiple integral doesn't look very friendly, and in fact in practice is very difficult to evaluate, even for the first few terms. The problem arises in the fact that each successive integration variable appears as one of the limits in the next integral in the sequence. It is, however, possible to transform the integral into a form that is a bit easier to manage. A detailed description of how this is done is given in Greiner & Reinhardt, section 8.3, so we'll summarize it here.

Consider the second order term in 27 and, for the moment, replace the  $-\infty$  lower limit by a finite limit, say  $t_0$ . Then we have

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) \quad (28)$$

The integral over  $t_2$  is taken between  $t_0$  (a constant) and an upper limit of  $t_1$  (the integration variable for the other integral). The other integral (over  $t_1$ ) is then taken from  $t_0$  up to the final time  $t$ . This two-dimensional integral is thus performed over a triangular area. If the region of integration is drawn in the  $t_1$ - $t_2$  plane (with  $t_1$  the horizontal axis and  $t_2$  the vertical, see Greiner & Reinhardt Fig 8.1), the area of integration is the triangle with lower edge  $t_2 = t_0$ , left-hand edge  $t_1 = t_0$  and upper edge  $t_1 = t_2$ .

We could swap the order of integration, and integrate over  $t_1$  first, then  $t_2$ . In this case we have

$$\int_{t_0}^t dt_2 \int_{t_2}^t dt_1 H_I(t_1) H_I(t_2) \quad (29)$$

Since  $t_1$  and  $t_2$  are both dummy indices, we can swap them in this last integral without changing anything, so we have

$$\int_{t_0}^t dt_2 \int_{t_2}^t dt_1 H_I(t_1) H_I(t_2) = \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 H_I(t_2) H_I(t_1) \quad (30)$$

Note that in 28,  $t_1 \geq t_2$  (the limits for the  $t_2$  integration are between  $t_0$  and  $t_1$ ) while in 30,  $t_1 \leq t_2$  (the limits for the  $t_2$  integration are now between  $t_1$  and  $t$ ). Also note that the order of the two Hamiltonian factors is always such that the time argument of the left-most Hamiltonian is larger than the time argument of the right-most Hamiltonian. Since both 28 and 30 represent the same integral, we can write

$$2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \quad (31)$$

$$\int_{t_0}^t dt_1 \int_{t_1}^t dt_2 H_I(t_2) H_I(t_1) \quad (32)$$

$$= \int_{t_0}^t dt_1 \left[ \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \int_{t_1}^t dt_2 H_I(t_2) H_I(t_1) \right] \quad (33)$$

$$= \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \mathcal{T} [H_I(t_1) H_I(t_2)] \quad (34)$$

where  $\mathcal{T}$  is the time-ordering operator.

This argument can be extended to the higher-order terms in the sum in 27. Basically, for the  $n$ th order term, there are  $n!$  ways of reordering the Hamiltonian factors so that their time arguments are strictly decreasing from left to right, so instead of the two terms in 33 we have  $n!$  terms. The result is L&P's eqn 5.33:

$$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \dots \int_{-\infty}^t dt_n \quad (35)$$

$$\mathcal{T} [H_I(t_1) H_I(t_2) \dots H_I(t_n)] \quad (36)$$

This is often written in a shorthand way as

$$U(t) = \mathcal{T} \left[ \exp \left( -i \int_{-\infty}^t dt' H_I(t') \right) \right] \quad (37)$$

Equation 36 might still not look particularly useful, as it involves a large number of (potentially infinite) integrals. To make matters worse, there is no guarantee that, for the Hamiltonians used to describe actual particle interactions, the series converges and, as L&P point out, in many cases it doesn't. The remarkable thing is that, provided that the terms in the sum asymptotically approach some limit, the first few terms in the series do actually give good agreement with scattering experiments.

#### PINGBACKS

Pingback: Wick's theorem for 2 fields

Pingback: S-matrix

Pingback: Wick's theorem - no equal time commutators