S-MATRIX

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 5.

The evolution operator $U(t)$ is used to follow the evolution of an initial state through an interaction and out the other side into a final state. It is written as a series over successively higher-order integrals:

$$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n \tag{1}$$

$$\mathscr{T}\left[H_I(t_1)\, H_I(t_2) \ldots H_I(t_n)\right] \tag{2}$$

where $H_I(t)$ is the interaction hamiltonian at time t. This is often written in the abbreviated form

$$U(t) = \mathscr{T}\left[\exp\left(-i \int_{-\infty}^{t} dt'\, H_I(t')\right)\right] \tag{3}$$

where the exponential is taken to represent the series in the first equation.

The S-matrix is defined as the limit of U at infinite upper time, that is

$$S \equiv \lim_{t\to\infty} U(t) = \mathscr{T}\left[\exp\left(-i \int_{-\infty}^{\infty} dt'\, H_I(t')\right)\right] \tag{4}$$

Since H_I is the total hamiltonian, it can be written as the integral of the hamiltonian density $\mathscr{H}_I(x)$ where x is now the spacetime four-vector. That is

$$H_I = \int d^3x\; \mathscr{H}_I(x) \tag{5}$$

If we combine the space integral with the time integral, we can write 4 as

$$S = \mathscr{T}\left[\exp\left(-i \int d^4x\; \mathscr{H}_I(x)\right)\right] \tag{6}$$

where now all four coordinates are integrated over all values. Since U is unitary, so is S:

$$S^{\dagger} = S^{-1} \tag{7}$$

$$S^{\dagger}S = 1 \tag{8}$$

The value of the S-matrix is that its matrix elements represent the amplitudes for the processes in which we start with some initial state $|i\rangle$ to a final state $|f\rangle$. In other words, if we know the S-matrix, we have a prediction for the probabilities of the various outcomes of particle experiments involving decay and scattering, so the S-matrix is central to the predictive ability of quantum field theory. In symbols, L&P show leading up to their eqn 5.39, that the amplitude for making a transition from the initial state $|i\rangle$ to the final state $|f\rangle$ is $\langle f|S|i\rangle$.

The catch, of course, is that it's far from easy to calculate S, since it involves calculating an infinite series of nested integrals, as in 2. In practice, we must specify the interaction hamiltonian (or, more usually, we begin with a Lagrangian and derive the hamiltonian from that), and then hope that the successive terms in 2 diminish fast enough that we can get a meaningful prediction using just the first few terms in the series. To calculate these terms, we note that hamiltonians are usually given as normal-ordered products, while the terms in 2 are time ordered, so we need a way of relating a time-ordered product to a normal-ordered product. This is given by Wick's theorem.

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