INTERACTIONS OF FIELDS - BASICS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 5, Problems 5.1 - 5.2.

Up to now, all the quantum field theory we've looked at has been for free fields, that is, for fields that do not interact with each other or with the environment. In ordinary quantum mechanics, this is equivalent to a Schrödinger equation without a potential term.

In field theory, a free field can be described by a pair of creation and annihilation operators, both of the same type. At each point in spacetime, a particle is annihilated and then re-created at the same point. Thus an operator pair of the form $a^{\dagger}a$ occurs in a free field system.

To introduce interactions (without which nothing in the universe would ever happen), we need terms that create and/or annihilate particles of different types or numbers. For a real scalar field, the general type of interaction can be represented by a Lagrangian of the form

$$\mathscr{L}=\mathscr{L}_{\text{free}}+\mathscr{L}_{\text{int}} \tag{1}$$

where $\mathscr{L}_{\text{free}}$ is the Lagrangian for the free field (containing quadratic terms in the field ϕ) and $\mathscr{L}_{\text{int}}$ is the interaction term, which could have terms of higher degree in the field, such as

$$\mathscr{L}_{\text{int}}=-\mu\phi^{3}-\lambda\phi^{4}-\sum_{n\ge5}\lambda_{(n)}\phi^{n} \tag{2}$$

Here μ and the λ terms are constants known as coupling constants.

The units of the coupling constants can be derived from the units of the Lagrangian which, in natural units are

$$[\mathscr{L}]=\left[\text{mass}^{4}\right] \tag{3}$$

The units of the field are, in 4-dimensional spacetime

$$[\phi]=\left[\text{mass}^{(4-2)/2}\right]=[\text{mass}] \tag{4}$$

Therefore, the units of $\lambda_{(n)}$ must be such that

$$\left[\text{mass}^4\right] = \left[\lambda_{(n)}\right]\left[\text{mass}^n\right] \tag{5}$$

so we have

$$\left[\lambda_{(n)}\right] = \left[\text{mass}^{4-n}\right] \tag{6}$$

In natural units, everything is expressed in mass units, so we can just say that the units of $\lambda_{(n)}$ are $4-n$.

In their equations 5.7 through 5.10, L&P give several examples of $\mathscr{L}_{\text{int}}$ terms involving both scalar and spinor fields. At this stage, the only constraint on these examples is that the Lagrangian be hermitian (or differ from a hermitian term by a total divergence). One example is:

$$\mathscr{L}_{\text{int}} = -h\overline{\psi}\psi\phi \tag{7}$$

where h is a coupling constant (a complex number, *not* Planck's constant!), ψ is a Dirac spinor and ϕ is a real scalar field. We also use

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 \tag{8}$$

In order for 7 to be hermitian, we must have

$$\mathscr{L}_{\text{int}}^\dagger = -h^*\phi^\dagger\psi^\dagger\left(\gamma^0\right)^\dagger\psi = \mathscr{L}_{\text{int}} \tag{9}$$

Using the identities

$$\left(\gamma^0\right)^2 = 1 \tag{10}$$

$$\left(\gamma^0\right)^\dagger = \gamma^0\gamma^0\gamma^0 = \gamma^0 \tag{11}$$

and, since ϕ is a real field, $\phi^\dagger = \phi$, we have

$$\mathscr{L}_{\text{int}}^\dagger = -h^*\phi^\dagger\psi^\dagger\gamma^0\psi \tag{12}$$

$$= -h^*\phi\overline{\psi}\psi \tag{13}$$

$$= -h^*\overline{\psi}\psi\phi \tag{14}$$

where the last line follows because a scalar field commutes with a spinor. Thus for 9 to be satisfied, we must have

$$h^* = h \tag{15}$$

so that h must be real.

Another example is

$$\mathscr{L}_{\text{int}} = -h'\overline{\psi}\gamma_5\psi\phi \tag{16}$$

In this case, we have

$$\mathscr{L}_{\text{int}}^{\dagger}=-h'^{*}\phi^{\dagger}\psi^{\dagger}\gamma_{5}^{\dagger}\left(\gamma^{0}\right)^{\dagger}\psi \tag{17}$$

From the definition of γ_5 and the identity $(\gamma^{\mu})^{\dagger}=\gamma^{0}\gamma^{\mu}\gamma^{0}$ and $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}$ we have

$$\gamma_{5}^{\dagger}=\left(i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}\right)^{\dagger} \tag{18}$$

$$=-i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \tag{19}$$

$$=-i\gamma^{0}\gamma^{3}\left(\gamma^{0}\right)^{2}\gamma^{2}\left(\gamma^{0}\right)^{2}\gamma^{1}\left(\gamma^{0}\right)^{2} \tag{20}$$

$$=-i\gamma^{0}\gamma^{3}\gamma^{2}\gamma^{1} \tag{21}$$

$$=i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \tag{22}$$

$$=\gamma_{5} \tag{23}$$

Therefore, since $\{\gamma_{5},\gamma^{\mu}\}=0$, we have

$$\mathscr{L}_{\text{int}}^{\dagger}=-h'^{*}\psi^{\dagger}\gamma_{5}\gamma^{0}\psi\phi \tag{24}$$

$$=h'^{*}\psi^{\dagger}\gamma^{0}\gamma_{5}\psi\phi \tag{25}$$

$$=h'^{*}\overline{\psi}\gamma_{5}\psi\phi \tag{26}$$

In order that

$$\mathscr{L}_{\text{int}}^{\dagger}=\mathscr{L}_{\text{int}} \tag{27}$$

we must therefore have

$$h'^{*}=-h' \tag{28}$$

so h' must be purely imaginary.

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