WICK'S THEOREM FOR 2 FIELDS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Section 5.2.

W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 8.

We've seen that, for interacting fields, the evolution operator may be written as the sum of multiple integrals over time-ordered products

$$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n \tag{1}$$

$$\mathscr{T}\left[H_I(t_1) H_I(t_2) \ldots H_I(t_n)\right] \tag{2}$$

where $H_I(t)$ is the time-dependent interaction Hamiltonian, defined as

$$H_I(t) \equiv U_0^{\dagger}(t) H_I U_0(t) \tag{3}$$

The two evolution operators are defined by their effects on the initial state $|\Psi_0(-\infty)\rangle$ which is assumed to be a linear combination of the free-particle basis states $|\beta\rangle$.

$$|\Psi_0(t)\rangle = U_0(t)|\Psi_0(-\infty)\rangle \equiv U_0(t)|i\rangle \tag{4}$$

$$|\Psi(t)\rangle = U_0(t) U(t) U_0^{\dagger}(t)|\Psi_0(t)\rangle \tag{5}$$

The problem with 2 is that Hamiltonians are usually given as normal-ordered products, in which all creation operators are moved to the left of all annihilation operators, so that a normal-ordered Hamiltonian gives zero when applied to the vacuum state $|0\rangle$. To solve this problem, we'd like a way of expressing a time-ordered product in terms of normal-ordered products. At first glance, this might seem to be impossible, since in a time-ordered product, we require both types of operators (creation and annihilation) to be ordered in increasing time from right to left, so naturally this can mix up the ordering of creation and annihilation operators, ruining any normal ordering that may have existed. However, it turns out that it is possible to express a time-ordered product as a sum of normal ordered products using a result known as *Wick's theorem*.

L&P give a detailed derivation of Wick's theorem for the case of two scalar fields in their equations 5.43 through 5.55. A detailed derivation for the cases of two and three fields (of either scalar of fermion type) is given in Greiner & Reinhardt, chapter 8. Here, we'll run through the derivation for fermion fields by following L&P's method. We'll call the fermion fields $\psi(x)$ and $\psi(x')$, with the usual reminder that these are 4-component spinors, whose components we can indicate by suffixes, so that $\psi_\alpha(x)$ is a component of ψ with $\alpha = 0,1,2,3$. Thus we need to do the calculation for individual components of the spinors. In what follows ψ_α is always a component of $\psi(x)$ and ψ_β is always a component of $\psi(x')$, so we'll drop the explicit dependence on the spacetime coordinate to simplify the notation.

For fermion fields, remember that time-ordering is defined as

$$\mathscr{T}\left[\psi_\alpha(x)\,\psi_\beta\left(x'\right)\right] \equiv \begin{cases} \psi_\alpha(x)\,\psi_\beta(x') & \text{if } t > t' \\ -\psi_\beta(x')\,\psi_\alpha(x) & \text{if } t' > t \end{cases} \tag{6}$$

The time-ordered product can be written as

$$\mathscr{T}\left[\psi_\alpha\psi_\beta\right] = \Theta\left(t-t'\right)\psi_\alpha\psi_\beta - \Theta\left(t'-t\right)\psi_\beta\psi_\alpha \tag{7}$$

where

$$\Theta\left(t-t'\right) = \begin{cases} 1 & \text{if } t > t' \\ \frac{1}{2} & \text{if } t = t' \\ 0 & \text{if } t < t' \end{cases} \tag{8}$$

The fermion field operator is expanded in terms of creation and annihilation operators according to

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2}\left(f_s(\mathbf{p})\,u_s(\mathbf{p})\,e^{-ip\cdot x} + \hat{f}_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\,e^{ip\cdot x}\right) \tag{9}$$

Thus each component ψ_α of ψ consists of the sum of a term containing only annihilation operators and a term containing only creation operators. We can write these as

$$\psi_{\alpha+} \equiv \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} f_s(\mathbf{p})\,u_s(\mathbf{p})\,e^{-ip\cdot x} \tag{10}$$

$$\psi_{\alpha-} \equiv \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \hat{f}_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\,e^{ip\cdot x} \tag{11}$$

In the product of two field operators, we can use the shorthand notation

$$\psi_\alpha \psi_\beta = \psi_{\alpha+}\psi_{\beta+} + \psi_{\alpha+}\psi_{\beta-} + \psi_{\alpha-}\psi_{\beta+} + \psi_{\alpha-}\psi_{\beta-} \tag{12}$$

In a normal-ordered product, all terms with a subscript containing a $-$ must lie to the left of all terms with a subscript containing a $+$. The only term in 12 that is not normal ordered is the second term: $\psi_{\alpha+}\psi_{\beta-}$. Normal ordering for fermion fields is done by assuming that the anticommutators of all fields are zero, so if we swap the order of the two fields in this term, we introduce a minus sign. That is

$$\begin{aligned} :\psi_\alpha \psi_\beta: &= \psi_{\alpha+}\psi_{\beta+} - \psi_{\beta-}\psi_{\alpha+} + \psi_{\alpha-}\psi_{\beta+} + \psi_{\alpha-}\psi_{\beta-} && (13)\\ &= \psi_\alpha \psi_\beta - \psi_{\alpha+}\psi_{\beta-} - \psi_{\beta-}\psi_{\alpha+} && (14)\\ &= \psi_\alpha \psi_\beta - \left\{\psi_{\alpha+}, \psi_{\beta-}\right\} && (15) \end{aligned}$$

Because of the anticommutators, we find that

$$\begin{aligned} :\psi_\beta \psi_\alpha: &= \psi_{\beta+}\psi_{\alpha+} - \psi_{\alpha-}\psi_{\beta+} + \psi_{\beta-}\psi_{\alpha+} + \psi_{\beta-}\psi_{\alpha-} && (16)\\ &= -\psi_{\alpha+}\psi_{\beta+} + \psi_{\beta-}\psi_{\alpha+} - \psi_{\alpha-}\psi_{\beta+} - \psi_{\alpha-}\psi_{\beta-} && (17)\\ &= -:\psi_\alpha \psi_\beta: && (18)\\ &= \psi_\beta \psi_\alpha - \left\{\psi_{\beta+}, \psi_{\alpha-}\right\} && (19) \end{aligned}$$

Since the application of any annihilation operator to the vacuum gives zero, we have

$$\psi_{\alpha+}|0\rangle = \langle 0|\psi_{\alpha-} = 0 \tag{20}$$

and likewise for ψ_β, so we get

$$\begin{aligned} \left\langle 0\middle|\left\{\psi_{\alpha+}, \psi_{\beta-}\right\}\middle|0\right\rangle &= \left\langle 0\middle|\psi_{\alpha+}\psi_{\beta-} + \psi_{\beta-}\psi_{\alpha+}\middle|0\right\rangle && (21)\\ &= \left\langle 0\middle|\psi_{\alpha+}\psi_{\beta-}\middle|0\right\rangle + 0 && (22)\\ &= \left\langle 0\middle|\psi_{\alpha+}\psi_{\beta-}\middle|0\right\rangle && (23)\\ &= \left\langle 0\middle|\psi_\alpha \psi_\beta\middle|0\right\rangle && (24) \end{aligned}$$

where the last line follows because the matrix elements of the other terms besides $\psi_{\alpha+}\psi_{\beta-}$ are all zero, due to 20. We also have

$$\left\{\psi_{\beta+}, \psi_{\alpha-}\right\} = \left\langle 0\middle|\psi_\beta \psi_\alpha\middle|0\right\rangle \tag{25}$$

Now the anticommutators are actually given by

$$\left\{f_r(\mathbf{p}), f_s^\dagger(\mathbf{q})\right\} = \left\{\hat{f}_r(\mathbf{p}), \hat{f}_s^\dagger(\mathbf{q})\right\} = \delta_{rs}\delta^3(\mathbf{p}-\mathbf{q}) \tag{26}$$

so they are just numbers, not operators. Therefore the matrix elements of the anticommutators are the same as the anticommutators themselves (assuming the states are normalized), so that

$$\left\langle 0\left|\psi_{\alpha}\psi_{\beta}\right|0\right\rangle =\left\{ \psi_{\alpha+},\psi_{\beta-}\right\} \tag{27}$$

Therefore, combining this with 15 we have

$$\psi_{\alpha}\psi_{\beta} =:\psi_{\alpha}\psi_{\beta}:+\left\{ \psi_{\alpha+},\psi_{\beta-}\right\} \tag{28}$$
$$=:\psi_{\alpha}\psi_{\beta}:+\left\langle 0\left|\psi_{\alpha}\psi_{\beta}\right|0\right\rangle \tag{29}$$
$$\psi_{\beta}\psi_{\alpha} =:\psi_{\beta}\psi_{\alpha}:+\left\{ \psi_{\beta+},\psi_{\alpha-}\right\} \tag{30}$$
$$=-:\psi_{\alpha}\psi_{\beta}:+\left\langle 0\left|\psi_{\beta}\psi_{\alpha}\right|0\right\rangle \tag{31}$$

From 7 we have

$$\mathscr{T}\left[\psi_{\alpha}\psi_{\beta}\right] =\Theta\left(t-t^{\prime}\right)\left(:\psi_{\alpha}\psi_{\beta}:+\left\langle 0\left|\psi_{\alpha}\psi_{\beta}\right|0\right\rangle \right)- \tag{32}$$
$$\Theta\left(t^{\prime}-t\right)\left(-:\psi_{\alpha}\psi_{\beta}:+\left\langle 0\left|\psi_{\beta}\psi_{\alpha}\right|0\right\rangle \right) \tag{33}$$
$$=:\psi_{\alpha}\psi_{\beta}:+\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}\psi_{\beta}\right]\right|0\right\rangle \tag{34}$$

where the last line uses 6. Restoring the spacetime arguments, this is

$$\mathscr{T}\left[\psi_{\alpha}\left(x\right)\psi_{\beta}\left(x^{\prime}\right)\right]=:\psi_{\alpha}\left(x\right)\psi_{\beta}\left(x^{\prime}\right):+\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}\left(x\right)\psi_{\beta}\left(x^{\prime}\right)\right]\right|0\right\rangle \tag{35}$$

The matrix element $\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}\psi_{\beta}\right]\right|0\right\rangle$ is just a number, and is known as a *Wick contraction*. It is usually written as

$$\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}\left(x\right)\psi_{\beta}\left(x^{\prime}\right)\right]\right|0\right\rangle \equiv\underbrace{\psi_{\alpha}\left(x\right)\psi}_{}{}_{\beta}\left(x^{\prime}\right) \tag{36}$$

We could go through the same argument for the case where one Dirac field is given by ψ and the other by $\overline{\psi}$ to get the result

$$\left\langle 0\left|\mathscr{T}\left[\psi_{\alpha}\left(x\right)\overline{\psi}_{\beta}\left(x^{\prime}\right)\right]\right|0\right\rangle \equiv\underbrace{\psi_{\alpha}\left(x\right)\overline{\psi}}_{}{}_{\beta}\left(x^{\prime}\right) \tag{37}$$

In this case, the Wick contraction is the same thing as i times the Feynman propagator:

$$iS_{F\alpha\beta}\left(x-x^{\prime}\right)=\underbrace{\psi_{\alpha}\left(x\right)\overline{\psi}}_{}{}_{\beta}\left(x^{\prime}\right) \tag{38}$$

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