WICK'S THEOREM - GENERAL CASE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 5, Exercise 5.4.

W. Greiner & J. Reinhardt, *Field Quantization*, Springer-Verlag (1996), Chapter 8, Exercise 8.2.

We've seen that the time-ordered product of two fields can be written in terms of a normal-ordered product and a contraction. This is a special case of Wick's theorem, applied to two fields:

$$\mathscr{T}\left[\psi_\alpha(x)\,\psi_\beta(x')\right] = :\psi_\alpha(x)\,\psi_\beta(x'): + \underbracket{\psi_\alpha(x)\,\psi}_\beta(x') \tag{1}$$

where the Wick contraction is a shorthand notation for the matrix element of a time-ordered product in the vacuum state:

$$\left\langle 0\left|\mathscr{T}\left[\psi_\alpha(x)\,\psi_\beta(x')\right]\right|0\right\rangle \equiv \underbracket{\psi_\alpha(x)\,\psi}_\beta(x') \tag{2}$$

In their exercise 5.4, L&P ask us to prove the general form of Wick's theorem using induction. The general theorem can be written as (here I'm using the notation in Greiner & Reinhardt (G&R), although I've left the hats off the operators, as all captial letters are operators here):

$$\begin{aligned}
\mathscr{T}[ABC\ldots XYZ] = {} & :ABC\ldots XYZ: + && (3)\\
& :\underbracket{AB}C\ldots XYZ: + && (4)\\
& :\underbracket{ABC}\ldots XYZ: + && (5)\\
& :ABC\ldots X\underbracket{YZ}: + && (6)\\
& :\underbracket{AB}\,\underbracket{CD}\ldots XYZ: + && (7)\\
& :ABCD\ldots XYZ: && (8)\\
& :ABC\ldots \underbracket{WX}\,\underbracket{YZ}: + && (9)\\
& \text{all higher order contractions} && (10)
\end{aligned}$$

The first line is the normal-ordered product of all the operators, with no contractions. The next 3 lines contain all possible single contractions between 2 operators, with the remaining uncontracted operators being normal-ordered.

Note that since a contraction is just a number (a matrix element), it can be taken outside the normal ordering. Also, for scalar field operators ϕ_1 and ϕ_2:

$$\mathscr{T}[\phi_1\phi_2] = \mathscr{T}[\phi_2\phi_1] \tag{11}$$

$$:\phi_1\phi_2: = :\phi_2\phi_1: \tag{12}$$

However, for fermion field operators ψ_1 and ψ_2, because of the anticommutation relations, we have

$$\mathscr{T}[\psi_1\psi_2] = -\mathscr{T}[\psi_2\psi_1] \tag{13}$$

$$:\psi_1\psi_2: = -:\psi_2\psi_1: \tag{14}$$

As a consequence of these relations, we have the general identity

$$:ABCDEF\ldots KLM\ldots: = \epsilon :ABF\ldots KM\ldots: \underbracket{CE}\,\underbracket{DL} \tag{15}$$

That is, we can take contractions of two operators that are separated by other operators in a normal product outside the normal product, but only if we commute the operators in the contraction through the other operators until they are adjacent. In doing these commutations, we need to factor in the appropriate sign change if the two operators being commuted are fermion operators. The factor ϵ on the RHS is thus ±1 depending on the number of fermion commutations we had to do.

We can see that 1 is a special case of 3 for two fields. The problem is to prove 3 by induction, using 1 as the anchor step. That is, we assume that 3 is true for n operators and then prove from this that it is true for $n+1$ operators.

I have to admit that with no more information than what is given in L&P, it would have taken me a long time (if ever) to arrive at such a proof. Fortunately, the proof given in G&R is quite clear, so I'll run through that here.

The first step is to prove a lemma. Given a set $A, B, \ldots, Y, Z$ of linear operators (G&R state that these are 'time-independent' but I don't see how they can be if we are to time-order them) with the conditions that

(1) The time argument of Z is smaller than that of all the other operators $A, B, \ldots, Y$.
(2) Z is a creation operator.
(3) All the other operators $A, B, \ldots, Y$ are annihilation operators.

Then we must show that

$$:AB\ldots XY:Z =:AB\ldots XYZ: + :\underbracket{AB\ldots XYZ}: + \ldots \tag{16}$$
$$:A\underbracket{B\ldots XYZ}: + \tag{17}$$
$$:AB\ldots X\underbracket{YZ}: \tag{18}$$

That is, if we take a normal ordered product of operators and multiply it on the right by another operator Z, the result is the normal ordered product of all the operators plus all possible contractions of the existing operators $A, B, \ldots, X, Y$ with the new operator Z.

First, we need to comment on a couple of the restrictions imposed above. With respect to assumption 2 (Z being a creation operator), suppose this isn't true, and that $Z = Z_+ + Z_-$ where Z_+ is an annihilation operator and Z_- is a creation operator. Then for Z_+ the LHS of 16 is automatically true, since normal ordering places annihilation operators on the right. The RHS, any contraction involving Z_+ will be of the form

$$\underbracket{AZ}_+ = \langle 0|\mathscr{T}[AZ_+]|0\rangle \tag{19}$$
$$= \langle 0|AZ_+|0\rangle \tag{20}$$
$$= 0 \tag{21}$$

The second line follows from the fact that $t_Z < t_A$ (assumption 1 above), which the last line follows because Z_+ is an annihilation operator acting on the vacuum, which is zero. Thus adding in an annihilation component to Z will make no difference to 16.

Now with respect to assumption 3 ($A, B, \ldots, Y$ are annihilation operators), again, suppose that some of these operators have a creation component. Then on the LHS of 16, all these components will lie to the left of the annihilation components. On the RHS, the extra contractions introduced will consist of contractions between a creation component of one of the operators, say O_-, with Z_- (since we know that Z can be taken to be a creation operator). That is, we have contractions of the form

$$\underbracket{O_- Z}_- = \langle 0|\mathscr{T}[O_-Z_-]|0\rangle \tag{22}$$
$$= \langle 0|O_-Z_-|0\rangle \tag{23}$$
$$= 0 \tag{24}$$

Again, the second line follows because $t_Z < t_O$, and the last line follows because we have a creation operator O_- acting to the left on a vacuum bra, which is always zero.

Thus the assumptions 2 and 3 above don't limit the generality of the proof, which we now derive.

To begin the induction, suppose that 16 is true for a product $BC\ldots XY$ of annihilation operators and a creation operator Z. That is, we assume that

$$:B\ldots XY\colon Z =:B\ldots XYZ\colon + :B\underbracket{C\ldots XYZ}\colon + \ldots \tag{25}$$

$$:B\ldots X\underbracket{YZ}\colon \tag{26}$$

Now multiply on the left by an annihilation operator A with a time greater than t_Z. We get

$$A:B\ldots XY\colon Z \;=\; A:B\ldots XYZ\colon + :AB\underbracket{C\ldots XYZ}\colon + \ldots \tag{27}$$

$$:AB\ldots X\underbracket{YZ}\colon \tag{28}$$

In the first term on the RHS, we can't put A inside the normal order because the operators as listed inside the normal ordering signs are *not* in normal order themselves (because Z is a creation operator and appears on the right). In the remaining terms, Z appears only within a contraction and is thus converted into an ordinary number, so in these terms it *is* valid to place A inside the normal ordering.

Now consider the first term on the RHS. We would like to show that

$$A:B\ldots XYZ\colon = :AB\ldots XYZ\colon + :\underbracket{ABC\ldots XYZ}\colon \tag{29}$$

That is, the term on the LHS is equal to a full normal ordering of all the operators plus the extra contraction of the new operator A with the creation operator Z.

First, we can commute the Z in the LHS so it lies at the start of the normal ordering. Depending on the presence of fermion operators, this might introduce a sign, so we'll multiply by $\epsilon = \pm 1$ to account for this:

$$A:B\ldots XYZ\colon = \epsilon AZBC\ldots XY \tag{30}$$

Note that the product on the RHS is *not* normal ordered, since the creation operator Z is still to the right of the new annihilation operator A. Now we can apply Wick's theorem for 2 fields 1 to the product AZ:

$$AZ = \mathscr{T}[AZ] = :AZ\colon + \underbracket{AZ} \tag{31}$$

Inserting this into 30 we have

$$A:B\ldots XYZ\colon = \epsilon :AZ\colon :BC\ldots Y\colon + \epsilon \underbracket{AZ} :BC\ldots Y\colon \tag{32}$$

In the first term, we have, since Z is the only creation operator so it must be leftmost in a normal ordering (and commuting it with A could introduce a minus sign):

$$\begin{aligned} :AZ::BC\ldots Y: &= \pm ZABC\ldots Y && (33)\\ &= \pm :ZABC\ldots Y: && (34) \end{aligned}$$

Now we can commute Z back to the right-hand end. Doing this will cancel the sign just introduced when it commutes with A. The remaining commutes with B through Y will introduce the factor ϵ again, so we have

$$\begin{aligned} \pm :ZABC\ldots Y: &=:AZBC\ldots Y: && (35)\\ &= \epsilon :ABC\ldots YZ: && (36) \end{aligned}$$

Inserting this into 32 we get

$$\begin{aligned} A:B\ldots XYZ: &= \epsilon^2 :ABC\ldots YZ: + \epsilon \underbracket{AZ} :BC\ldots Y: && (37)\\ &=:ABC\ldots YZ: + \epsilon \underbracket{AZ} :BC\ldots Y: && (38) \end{aligned}$$

since $\epsilon^2 = (\pm 1)^2 = 1$.

For the last term, we need to commute the Z in the contraction $\underbracket{AZ}$ back to the right-hand end of the product. We can use 15 to do this, and we see that a factor of $\epsilon = \pm 1$ gets introduced as the Z commutes through the product. The crucial point is that this ϵ is the same factor as that in 38 since it is obtained by the same sequence of commutations. Therefore

$$\underbracket{AZ} :BC\ldots Y: = \epsilon :\underbracket{ABC\ldots YZ}: \tag{39}$$

Placing this in 38 we get finally

$$\begin{aligned} A:B\ldots XYZ: &=:ABC\ldots YZ: + \epsilon^2 :\underbracket{ABC\ldots YZ}: && (40)\\ &=:ABC\ldots YZ: + :\underbracket{ABC\ldots YZ}: && (41) \end{aligned}$$

This matches equation 29, so the lemma 16 is now proved.

Notice that the same proof applies if some of the operators in the original product in the LHS of 16 are contracted. These contractions are just numbers and can be extracted from the product of the remaining operators using 15 (possibly with a factor of $\epsilon = \pm 1$). The same steps can be followed and the contractions reinserted into the correct order at the end (generating another factor of ϵ which cancels the first one).

From here, it's a fairly short step to the general Wick's theorem. We know that it is true for two fields, as in 1 and that this is a special case of the proposed general formula 3. Therefore, let's assume that 3 is true for a time-ordered product $AB\ldots XY$ and then multiply the result by a new operator Z on the right. That is we start by assuming that

$$\mathscr{T}[AB\ldots XY] = :AB\ldots XY: + :\underbracket{AB}\ldots XY: + \tag{42}$$

$$+ :\underbracket{AB}\underbracket{CD}\ldots XY: + \ldots \tag{43}$$

By assumption 1 above, the time t_Z is less than the times of all the other operators, so for the LHS we have

$$\mathscr{T}[AB\ldots XY]Z = \mathscr{T}[AB\ldots XYZ] \tag{44}$$

On the RHS, we can apply the lemma to each term in the expansion. In doing this, we see that for each term on the RHS of 42, multiplying on the right by Z will produce a sum of terms consisting of a normal ordered product of all uncontracted operators, plus a sum of terms which consist of a normal ordered product in which each operator that was uncontracted in the original term is contracted, in turn, with the new operator Z. Thus the final result is the general Wick's theorem 3.

As G&R point out at the end of their proof, assumption 1 (that t_Z is the smallest time) above doesn't affect the generality of the proof, since if the original operators are not in decreasing order of time, we can permute them on both sides of the equation so that they are. This introduces the same sign factor ϵ on both sides, after which we can follow the same steps in the proof. At the end of the calculation, we can reverse the permutation so that the operators are in their original order, which introduces another factor of ϵ, cancelling the first one.

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