SCALAR-SCALAR SCATTERING IN THE YUKAWA INTERACTION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 6, Exercise 6.1.

Thanks to contributors on Physics Forums for help with clarifying some points in this post.

As a first application of Wick's theorem, we'll look at a scattering problem using the Yukawa interaction, for which the interaction term in the Lagrangian looks like

$$\mathscr{L}_{\text{int}} = -h\overline{\psi}\psi\phi \tag{1}$$

where ψ is a fermion field and ϕ is a real scalar (boson) field.

According to the usual recipe for deriving the Hamiltonian from the Lagrangian, this gives rise to an interaction Hamiltonian of

$$\mathscr{H}_I = \pi(x)\dot{\phi}(x) - \mathscr{L}_{\text{int}} = h :\overline{\psi}\psi\phi: \tag{2}$$

which follows because the interaction Lagrangian doesn't have any derivative terms. Note we've applied normal ordering to the Hamiltonian to give zero energy in the vacuum state.

The scattering problem we wish to study is the scattering of two bosons:

$$B + B \to B + B \tag{3}$$

The idea is to work out the elements of the S-matrix between the initial and final states. The initial state consists of two bosons, each with some momentum, and the final state also consists of two bosons, each with its own momentum, subject to the constraint that the total momentum is conserved throughout the process.

If the scattering takes place via the Yukawa interaction, then the S-matrix is given by

$$S = \mathscr{T}\left[\exp\left(-i\int d^4x\, \mathscr{H}_I(x)\right)\right] \tag{4}$$

which is expanded into a series in terms of normal ordered products using Wick's theorem

$$\mathscr{T}\left[ABC\ldots XYZ\right] = :ABC\ldots XYZ: + \tag{5}$$
$$:\underbracket{AB}C\ldots XYZ: + \tag{6}$$
$$:\underbracket{ABC}\ldots XYZ: + \tag{7}$$
$$:ABC\ldots X\underbracket{YZ}: + \tag{8}$$
$$:\underbracket{AB}\,\underbracket{CD}\ldots XYZ: + \tag{9}$$
$$:ABCD\ldots XYZ: \tag{10}$$
$$:ABC\ldots \underbracket{WX}\,\underbracket{YZ}: + \tag{11}$$
$$\text{all higher order contractions} \tag{12}$$

The first step is to work out what terms in this expansion will give non-zero contributions to the result. The S-matrix describes the transition from an initial state $|i\rangle$ to a final state $|f\rangle$ by the matrix element $\langle f|S|i\rangle$. In our case, both the initial and final states are given by (I'm ignoring the dependence on momentum here) $|BB\rangle$. The scattering process is modelled by annihilating the two initial bosons, then applying the interaction, and finally creating the two bosons that emerge from the scattering. Since no fermions appear in either the initial or final states, there can be no terms in the expansion that contain naked fermion operators, because acting on $|BB\rangle$ with a fermion annihilator or on $\langle BB|$ with a fermion creator both give zero. As a result, all occurrences of $\overline{\psi}$ must be contracted with a corresponding ψ in the Wick expansion.

What about the bosons? The scalar field is given as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a(p)\,e^{-ip\cdot x} + a^{\dagger}(p)\,e^{ip\cdot x}\right) \tag{13}$$

We can write this as

$$\phi(x) = \phi_+(x) + \phi_-(x) \tag{14}$$

where ϕ_+ contains the annihilation operators $a(p)$ and ϕ_- contains the creation operators $a^{\dagger}(p)$. To describe the scattering, we want to begin by annihilating the two incoming bosons, so we apply ϕ_+ twice. After the interaction, we need to create the two outgoing bosons, so we apply ϕ_- twice. We therefore need 4 applications of the ϕ field, so the lowest order term in the Wick expansion that gives a non-zero result is

$$\mathscr{T}\left[:\left(\overline{\psi}\psi\phi\right):_{x_1}:\left(\overline{\psi}\psi\phi\right):_{x_2}:\left(\overline{\psi}\psi\phi\right):_{x_3}:\left(\overline{\psi}\psi\phi\right):_{x_4}\right]^{(4)} \tag{15}$$

where the spacetime coordinates are given as subscripts x_i to save space.

We now apply Wick's theorem to get the expansion in terms of normal ordered products. We want to contract all the $\overline{\psi}$ and ψ factors in all ways that are unique up to a permutation of the spacetime coordinates. In doing this, we first recall that no contractions are done between fields at equal times. To satisfy these constraints, we see that we can involve only an even number of time points in each set of contractions. If we tried to, say, contract the fields between 3 or the 4 time points, then the fourth time point would be left on its own and would need to contract with itself, which isn't allowed.

The first possibility is

$$:(\overline{\psi}\psi\phi)_{x_1}(\overline{\psi}\psi\phi)_{x_2}(\overline{\psi}\psi\phi)_{x_3}(\overline{\psi}\psi\phi)_{x_4}: \tag{16}$$

In this case, the first contractions are done between the time points $t_1$and t_2, which corresponds to the time coordinates of the outgoing bosons. The second set of contractions are done between t_3 and t_4, the times of the incoming bosons. There are no other permutations of these coordinates that satisfy these constraints, so this is unique.

A second possibility is

$$:(\overline{\psi}\psi\phi)_{x_1}(\overline{\psi}\psi\phi)_{x_2}(\overline{\psi}\psi\phi)_{x_3}(\overline{\psi}\psi\phi)_{x_4}: \tag{17}$$

This links a time t_1 of an outgoing boson with t_3 of an incoming boson, and likewise for t_2 with t_4. The same result occurs if we link t_1 with t_4 and t_2 with t_3.

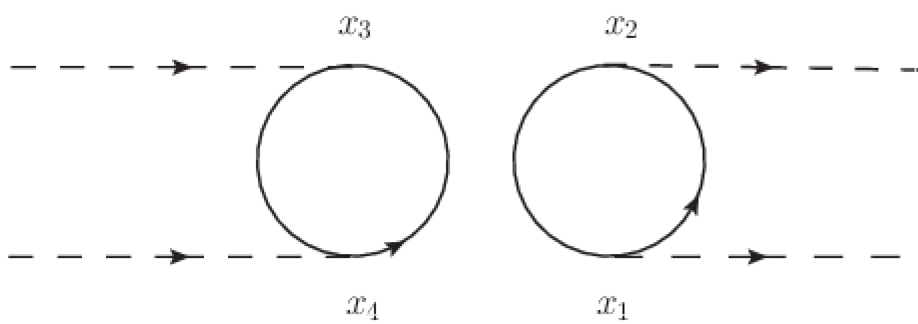
Finally, we can link all four times in a loop:

$$:(\overline{\psi}\psi\phi)_{x_1}(\overline{\psi}\psi\phi)_{x_2}(\overline{\psi}\psi\phi)_{x_3}(\overline{\psi}\psi\phi)_{x_4}: \tag{18}$$

Each of these contractions is equivalent to a Feynman propagator. Recall that since the ψs are spinors, they each consist of 4 components, so each contraction is really between a pair of components:

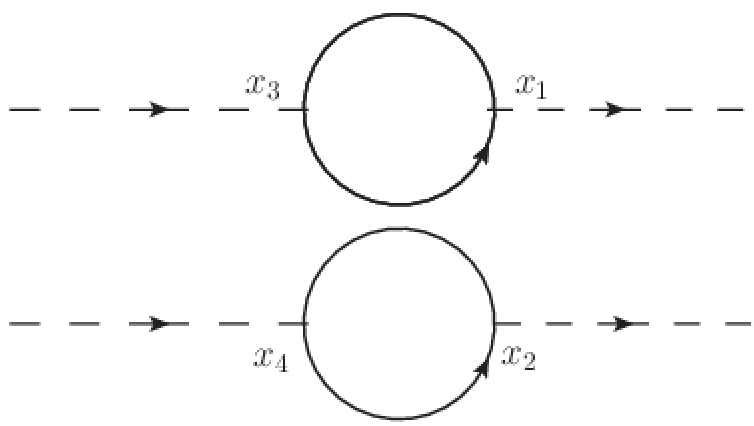
$$\left\langle 0\right|\mathscr{T}\left[\psi_\alpha(x)\overline{\psi}_\beta(x')\right]\left|0\right\rangle \equiv \psi_\alpha(x)\overline{\psi}_\beta(x') \tag{19}$$

We can display these terms in the Wick expansion using Feynman diagrams. The diagram for 16 is



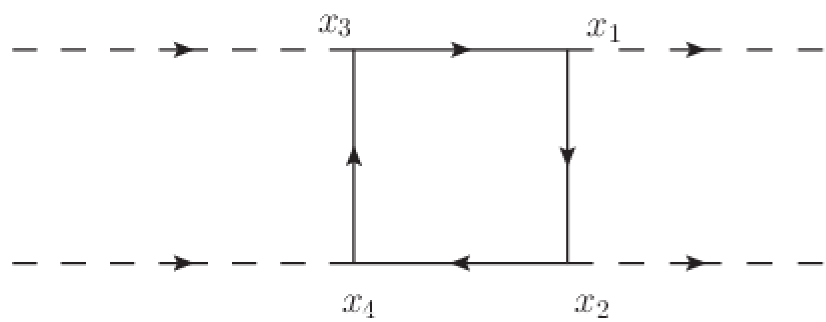
Time flows left to right, so the incoming bosons are shown as dashed lines moving in to points x_3 and x_4. The fermion propagators form a loop from x_3 to x_4 and back again from x_4 to x_3. On the right, we start with a fermion loop between x_1 and x_2 and end with the two bosons leaving the interaction.

The diagram for 17 is



This time, each fermion loop connects an incoming boson to an outgoing boson.

Finally, the diagram for 18 is



The four fermion propagators form a box connecting both incoming bosons to both outgoing bosons.

The first two diagrams are *disconnected* graphs, in that the scattering process consists of two separate sections. The last diagram shows a connected graph. It seems that only connected graphs contribute to the scattering process, although I'm still hazy on this point so we'll need to wait till we progress a bit more into the subject.

At the moment, I'm not sure the first box represents anything real, as there is no path from the incoming to the outgoing particles. The second graph might indicate no scattering at all, as the two bosons come in and go out without actually interacting with each other. The last diagram shows a genuine interaction taking place between the two bosons.

Keep in mind that these are only the lowest order terms in the Wick expansion, so more complex processes may occur in higher order terms.

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