ELECTRON-ELECTRON SCATTERING IN THE YUKAWA INTERACTION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 6, Exercise 6.5.

As another example of calculating an S-matrix element, L&P study electron-electron scattering in their section 6.4. The process is

$$e^-(p_1)+e^-(p_2)\to e^-\left(p_1'\right)+e^-\left(p_2'\right) \tag{1}$$

The Yukawa hamiltonian is

$$\mathscr{H}_I = h\,{:}\overline{\psi}\psi\phi{:} \tag{2}$$

Since there are no bosons in either the initial or final state, all boson field operators ϕ must be contracted in all terms of the Wick expansion. Since each contraction removes two fields, this requires there to be an even number of ϕ fields in each term of the expansion, so only even order terms will give non-zero results.

L&P go through the calculation of the second order term in the expansion. Their derivation is for the most part quite clear, so I'll summarize the argument here and fill in a few gaps.

The second order term is

$$S^{(2)}=\frac{(-ih)^2}{2!}\int d^4x_1\int d^4x_2\,\mathscr{T}\left[{:}\left(\overline{\psi}\psi\phi\right)_{x_1}{:}{:}\left(\overline{\psi}\psi\phi\right)_{x_2}{:}\right] \tag{3}$$

After applying Wick's theorem and isolating the terms that give a non-zero contribution to the S-matrix element, we get their equation 6.37:

$$\begin{aligned}S^{(2)}=&\frac{(-ih)^2}{2!}\int d^4x_1\int d^4x_2\, i\Delta_F(x_1-x_2)\\&\times\left\langle e^-\left(\mathbf{p}_1'\right)e^-\left(\mathbf{p}_2'\right)\left|\overline{\psi}_-^{\alpha}(x_1)\,\overline{\psi}_-^{\beta}(x_2)\,\psi_+^{\alpha}(x_1)\,\psi_+^{\beta}(x_2)\right|e^-(\mathbf{p}_1)\,e^-(\mathbf{p}_2)\right\rangle\end{aligned} \tag{4}$$

The indexes α and β indicate spinor components and are summed.

Now for the initial state we can use the result

$$\left|e^{-}\left(\mathbf{p},s\right)\right\rangle \equiv\sqrt{\frac{\left(2\pi\right)^{3}}{V}}f_{s}^{\dagger}\left(\mathbf{p}\right)\left|0\right\rangle \tag{5}$$

The initial state, before scattering, is then (ignoring the spin):

$$\left|e^{-}\left(\mathbf{p}_{1}\right)e^{-}\left(\mathbf{p}_{2}\right)\right\rangle =\frac{\left(2\pi\right)^{3}}{V}f_{s}^{\dagger}\left(\mathbf{p}_{1}\right)f_{s}^{\dagger}\left(\mathbf{p}_{2}\right)\left|0\right\rangle \tag{6}$$

The Fourier decomposition of the field gives us (again, ignoring spin - I'm assuming L&P take one spin and use that throughout):

$$\psi_{+}^{\alpha}\left(x\right)=\int\frac{d^{3}k}{\sqrt{2\left(2\pi\right)^{3}E_{k}}}f\left(\mathbf{k}\right)u^{\alpha}\left(\mathbf{k}\right)e^{-ik\cdot x} \tag{7}$$

Applying this twice to 6 we get L&P's equation 6.39:

$$\begin{aligned}\psi_{+}^{\alpha}\left(x_{1}\right)\psi_{+}^{\beta}\left(x_{2}\right)\left|e^{-}\left(\mathbf{p}_{1}\right)e^{-}\left(\mathbf{p}_{2}\right)\right\rangle = & \int\frac{d^{3}k}{\sqrt{2\left(2\pi\right)^{3}E_{k}}}\int\frac{d^{3}k'}{\sqrt{2\left(2\pi\right)^{3}E_{k'}}}\\ & \times u^{\alpha}\left(k\right)u^{\beta}\left(k'\right)e^{-ik\cdot x_{1}}e^{-ik'\cdot x_{2}}f\left(\mathbf{k}\right)f\left(\mathbf{k}'\right)\\ & \times\frac{\left(2\pi\right)^{3}}{V}f_{s}^{\dagger}\left(\mathbf{p}_{1}\right)f_{s}^{\dagger}\left(\mathbf{p}_{2}\right)\left|0\right\rangle \qquad (8)\\ = & \int\frac{d^{3}k}{\sqrt{2E_{k}V}}\frac{d^{3}k'}{\sqrt{2E_{k'}V}}u^{\alpha}\left(k\right)u^{\beta}\left(k'\right)e^{-ik\cdot x_{1}}e^{-ik'\cdot x_{2}}\qquad (9)\\ & \times f\left(\mathbf{k}\right)f\left(\mathbf{k}'\right)f_{s}^{\dagger}\left(\mathbf{p}_{1}\right)f_{s}^{\dagger}\left(\mathbf{p}_{2}\right)\left|0\right\rangle \qquad (10)\end{aligned}$$

By using anticommutators, we can reduce the operator product $f\left(\mathbf{k}\right)f\left(\mathbf{k}'\right)f_{s}^{\dagger}\left(\mathbf{p}_{1}\right)f_{s}^{\dagger}\left(\mathbf{p}_{2}\right)$ to a sum of two terms involving delta functions over the momenta, which can then be integrated to give L&P's equation 6.43:

$$\begin{aligned}\psi_{+}^{\alpha}\left(x_{1}\right)\psi_{+}^{\beta}\left(x_{2}\right)\left|e^{-}\left(\mathbf{p}_{1}\right)e^{-}\left(\mathbf{p}_{2}\right)\right\rangle = & \frac{1}{\sqrt{2E_{1}V}}\frac{1}{\sqrt{2E_{2}V}}\\ & \times\Big[u^{\alpha}\left(p_{1}\right)u^{\beta}\left(p_{2}\right)e^{-ip_{1}\cdot x_{1}}e^{-ip_{2}\cdot x_{2}}\\ & -u^{\alpha}\left(p_{2}\right)u^{\beta}\left(p_{1}\right)e^{-ip_{2}\cdot x_{1}}e^{-ip_{1}\cdot x_{2}}\Big]\left|0\right\rangle \end{aligned} \tag{11}$$

By similar reasoning we can get the expression for the final state, then convert the propagator $\Delta_{F}\left(x_{1}-x_{2}\right)$ to its Fourier transform, and then combine the two to get the S-matrix element given in L&P's equation 6.46:

$$S_{fi}^{(2)} = -\frac{(-ih)^2}{2!}\int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4}\Delta_F(q)\, e^{iq\cdot(x_1-x_2)}$$
$$\times\left[\overline{u}^\alpha(p_1')\,\overline{u}^\beta(p_2')\,e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2} - \overline{u}^\alpha(p_2')\,\overline{u}^\beta(p_1')\,e^{ip_2'\cdot x_1}e^{ip_1'\cdot x_2}\right]$$
$$\times\left[u^\alpha(p_1)\,u^\beta(p_2)\,e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2} - u^\alpha(p_2)\,u^\beta(p_1)\,e^{-ip_2\cdot x_1}e^{-ip_1\cdot x_2}\right]$$
$$\times\frac{1}{\sqrt{2E_1V}}\frac{1}{\sqrt{2E_2V}}\frac{1}{\sqrt{2E_1'V}}\frac{1}{\sqrt{2E_2'V}} \tag{12}$$

The key to simplifying this (a bit, at least) is to note that x_1 and x_2 are dummy variables (since they are integrated) and α and β are dummy indexes (since they are summed). We can isolate the terms in brackets:

$$\textcircled{A} \equiv \overline{u}^\alpha(p_1')\,\overline{u}^\beta(p_2')\,e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2} \tag{13}$$
$$\textcircled{B} \equiv \overline{u}^\alpha(p_2')\,\overline{u}^\beta(p_1')\,e^{ip_2'\cdot x_1}e^{ip_1'\cdot x_2} \tag{14}$$
$$\textcircled{C} \equiv u^\alpha(p_1)\,u^\beta(p_2)\,e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2} \tag{15}$$
$$\textcircled{D} \equiv u^\alpha(p_2)\,u^\beta(p_1)\,e^{-ip_2\cdot x_1}e^{-ip_1\cdot x_2} \tag{16}$$

We have

$$\textcircled{A}\times\textcircled{D} = \left(\overline{u}^\alpha(p_1')\,\overline{u}^\beta(p_2')\,e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2}\right)$$
$$\times\left(u^\alpha(p_2)\,u^\beta(p_1)\,e^{-ip_2\cdot x_1}e^{-ip_1\cdot x_2}\right) \tag{17}$$

If we swap $\alpha \leftrightarrow \beta$ and $x_1 \leftrightarrow x_2$ we get $\textcircled{B}\times\textcircled{C}$. Similarly, if we start with $\textcircled{B}\times\textcircled{D}$ and swap $\alpha \leftrightarrow \beta$ and $x_1 \leftrightarrow x_2$ we get $\textcircled{A}\times\textcircled{C}$. Therefore we have

$$-\int d^4x_1 \int d^4x_2 \left(\textcircled{A}-\textcircled{B}\right)\left(\textcircled{C}-\textcircled{D}\right) = -\int d^4x_1 \int d^4x_2 \left(2\textcircled{A}\times\textcircled{C} - 2\textcircled{B}\times\textcircled{C}\right) \tag{18}$$
$$= 2\int d^4x_1 \int d^4x_2 \textcircled{C}\left(\textcircled{B}-\textcircled{A}\right) \tag{19}$$

Inserting this into 12, we get L&P's equation 6.47:

$$\begin{aligned}S_{fi}^{(2)} &= (-ih)^2 \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) e^{iq\cdot(x_1-x_2)} \\ &\times \left[\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') e^{ip_2'\cdot x_1} e^{ip_1'\cdot x_2} - \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') e^{ip_1'\cdot x_1} e^{ip_2'\cdot x_2}\right] \\ &\times u^\alpha(p_1) u^\beta(p_2) e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} \frac{1}{\sqrt{2E_1V}}\frac{1}{\sqrt{2E_2V}}\frac{1}{\sqrt{2E_1'V}}\frac{1}{\sqrt{2E_2'V}}\end{aligned} \tag{20}$$

As x_1 and x_2 now appear only in exponentials, we can integrate them to get delta functions. We have

$$\begin{aligned}S_{fi}^{(2)} &= (-ih)^2 (2\pi)^4 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} \Delta_F(q) e^{-iq\cdot x_2} \\ &\times \left[\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') \delta^4(-q-p_2'+p_1) e^{ip_1'\cdot x_2} - \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') \delta^4(-q-p_1'+p_1) e^{ip_2'\cdot x_2}\right] \\ &\times u^\alpha(p_1) u^\beta(p_2) e^{-ip_2\cdot x_2} \frac{1}{\sqrt{2E_1V}}\frac{1}{\sqrt{2E_2V}}\frac{1}{\sqrt{2E_1'V}}\frac{1}{\sqrt{2E_2'V}} \qquad (21) \\ &= (-ih)^2 (2\pi)^4 \int d^4q\, \Delta_F(q) \qquad (22) \\ &\times \left[\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') \delta^4(-q-p_2'+p_1)\delta^4(q-p_1'+p_2)\right. \qquad (23) \\ &\left. - \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') \delta^4(-q-p_1'+p_1)\delta^4(q-p_2'+p_2)\right] \qquad (24) \\ &\times u^\alpha(p_1) u^\beta(p_2) \frac{1}{\sqrt{2E_1V}}\frac{1}{\sqrt{2E_2V}}\frac{1}{\sqrt{2E_1'V}}\frac{1}{\sqrt{2E_2'V}} \qquad (25)\end{aligned}$$

The two delta function products are actually equivalent. For the first one

$$\begin{aligned}\delta^4(-q-p_2'+p_1) &\implies q = p_1 - p_2' \qquad (26) \\ \delta^4(q-p_1'+p_2) &\implies p_1 - p_2' = p_1' - p_2 \qquad (27) \\ &\implies p_1 + p_2 = p_1' + p_2' \qquad (28)\end{aligned}$$

which is just momentum conservation. A similar calculation shows that the second product leads to the same condition. We can therefore replace a delta function in each product by $\delta^4(p_1+p_2-p_1'-p_2')$ which is independent of q so can be taken outside the integral, and integrate over the other one. This gives

$$S_{fi}^{(2)} = (-ih)^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_1' - p_2'\right) \int d^4q \, \Delta_F(q)$$
$$\times \left[\overline{u}^\alpha \left(p_2'\right) \overline{u}^\beta \left(p_1'\right) \delta^4 \left(-q - p_2' + p_1\right) - \overline{u}^\alpha \left(p_1'\right) \overline{u}^\beta \left(p_2'\right) \delta^4 \left(-q - p_1' + p_1\right)\right]$$
$$\times u^\alpha (p_1) u^\beta (p_2) \frac{1}{\sqrt{2E_1V}} \frac{1}{\sqrt{2E_2V}} \frac{1}{\sqrt{2E_1'V}} \frac{1}{\sqrt{2E_2'V}} \tag{29}$$
$$= (-ih)^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_1' - p_2'\right) \tag{30}$$
$$\times \left[\overline{u}^\alpha \left(p_2'\right) \overline{u}^\beta \left(p_1'\right) \Delta_F \left(p_1 - p_2'\right) - \overline{u}^\alpha \left(p_1'\right) \overline{u}^\beta \left(p_2'\right) \Delta_F \left(p_1 - p_1'\right)\right] \tag{31}$$
$$\times u^\alpha (p_1) u^\beta (p_2) \frac{1}{\sqrt{2E_1V}} \frac{1}{\sqrt{2E_2V}} \frac{1}{\sqrt{2E_1'V}} \frac{1}{\sqrt{2E_2'V}} \tag{32}$$

Note that the result consists of two terms, the second of which can be obtained from the first by swapping $p_1' \leftrightarrow p_2'$ and changing the sign. This is due to fact that swapping fermions also swaps the sign.

PINGBACKS

Pingback: Fermion scattering - fourth-order Feynman diagrams