FEYNMAN AMPLITUDE & RULES: VERTICES, EDGES AND LOOPS

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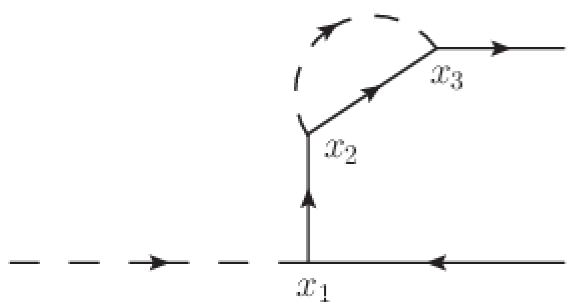
Post date: 6 Aug 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 6, Exercise 6.6.

In L&P's section 6.6, they discuss several rules for converting Feynman diagrams to S-matrix elements. The one we're interested in here is how to determine the overall factor of 2π. As an example, consider the diagram for the decay of a boson into an electron and positron, given as L&P's Figure 6.2c:



The S-matrix element, in the form after the Wick contractions have been converted to propagators but before any integrations over coordinates is

$$S^{(3c)} = (-ih)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4}$$
$$\times i\Delta_F(q_1) e^{-iq_1\cdot(x_2-x_3)} iS_{F\beta\alpha}(q_2) e^{-iq_2\cdot(x_2-x_1)} iS_{F\gamma\beta}(q_3) e^{-iq_3\cdot(x_3-x_2)}$$
$$\times \frac{1}{\sqrt{2\omega_k V}} \frac{1}{\sqrt{2E_p V}} \frac{1}{\sqrt{2E_{p'} V}} e^{-ik\cdot x_1} v_{s'\alpha}(p') e^{ip'\cdot x_1} \overline{u}_{s\gamma}(p) e^{ip\cdot x_3} \quad (1)$$

There is one factor of $(2\pi)^{-4}$ for each propagator, which means one factor of $(2\pi)^{-4}$ for each internal line (that is, a line with both ends anchored to vertices within the diagram). When we integrate over a coordinate such as x_1, we get $(2\pi)^4$ times a delta function, so there will be a factor of $(2\pi)^4$

for each vertex in the diagram. In our calculation of this matrix element, we extracted one delta function which imposed momentum conservation and kept a factor of $(2\pi)^4$ associated with this delta function. The final result was

$$S^{(3c)} = \frac{(-ih)^3 (2\pi)^4}{\sqrt{2\omega_k V}\sqrt{2E_p V}\sqrt{2E_{p'} V}} \delta^4 \left(k - p' - p\right) \int \frac{d^4 q}{(2\pi)^4} \\ \times i\Delta_F(q)\, \overline{u}_s(p)\, iS_F(p-q)\, iS_F(p)\, v_{s'}\left(p'\right) \tag{2}$$

One factor of $(2\pi)^{-4}$ is left inside the integral. This is the usual convention.

The parts of the S-matrix element, by convention, are written as

$$S_{fi}^{(j)} = i(2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f\right) \prod_i \frac{1}{\sqrt{2E_i V}} \prod_f \frac{1}{\sqrt{2E_f V}} \mathscr{M}_{fi}^{(i)} \tag{3}$$

where the superscript (i) indicates a particular term in the Wick expansion.

That is, we extract the momentum conserving delta function with its factor of $(2\pi)^4$ and the energy/volume factors and multiply them by what is known as the *Feynman amplitude* $\mathscr{M}_{fi}$. The Feynman amplitude is where the interaction happens. For the S-matrix element 2 we have

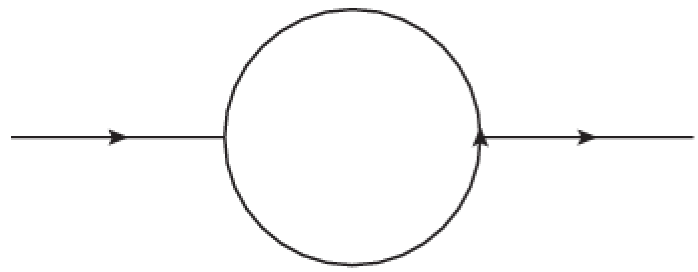
$$i\mathscr{M}_{fi}^{(3c)} = (-ih)^3 \int \frac{d^4 q}{(2\pi)^4} i\Delta_F(q)\, \overline{u}_s(p)\, iS_F(p-q)\, iS_F(p)\, v_{s'}\left(p'\right) \tag{4}$$

From the above discussion, we can see that the factor of 2π that appears in a Feynman amplitude is given by $(2\pi)^{4(v-n-1)}$ where v is the number of vertices in the diagram and n is the number of *internal* lines. However, by looking at the Feynman diagram above, we see that $v-n-1$ is also the negative of the number of loops ℓ in the diagram (which is 1 in this case). In fact, it is true in general that

$$v - n - 1 = -\ell \tag{5}$$

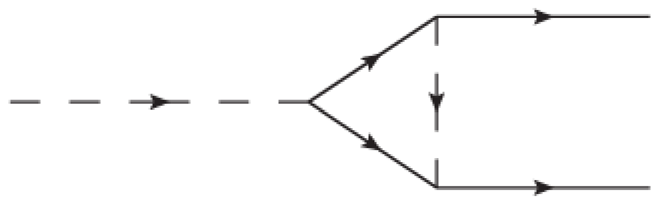
This is really a theorem in graph theory rather than physics, and we can prove it by induction. First, we should note that the term 'loop' as used here should really be called a 'cycle' in graph theory, as a loop in graph theory is a single edge that loops back to its starting point, while a cycle is a path through several edges that returns to its starting point.

We start by noting that the theorem is true for the simplest case:



The upper and lower semicircles each count as an internal edge, so $v = n = 2$ and $v - n - 1 = -1$ which is correct for a single loop.

Now consider this diagram:

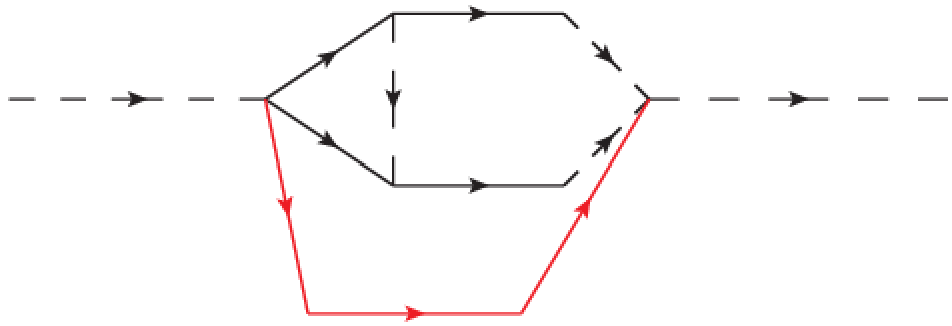


Here we have $v = n = 3$ and $\ell = 1$ so again $v - n - 1 = -\ell$. We can see from this that if we have a polygon with v vertices, it will also have $n = v$ edges and form a single loop, so the theorem is true for any single polygon of any size. It will also be true if we have any number N of separate polygons, none of which share common edges, since for polygon i, $v_i - n_i - 1 = -1$, so adding up all N polygons

$$\sum_{i=1}^{N}(v_i - n_i - 1) = \sum_{i=1}^{N}(v_i - n_i) - N = -N = -\ell \qquad (6)$$

since for polygon i, $v_i = n_i$, and if we have N separate polygons, we also have N loops.

Now suppose we apply induction by assuming that for some diagram the theorem is true (we've seen it's true for a single loop). We now add another loop and wish to show that the theorem is still true for the enlarged diagram. The situation is illustrated as follows.



The black lines indicate the starting diagram. We can see that for the black diagram, $v=6$ and $n=7$ so $v-n-1=-2$, which is correct as there are 2 loops.

Now we add another loop by attaching the red path. The new polygon ($(N+1)$th polygon) consisting of the red edges and the lower 3 black edges contains 6 vertices and 6 edges. However, 4 of these vertices and 3 of these edges are common edges between the new polygon and the existing polygons, so we don't count these again. In general, if we add a polygon by attaching it to an existing diagram, this new polygon will share e edges and $e+1$ vertices. Thus the net contribution from the new polygon with v_{N+1} vertices and n_{N+1} edges is $v_{N+1}-e-1$ vertices and $n_{N+1}-e$ edges. Since $v_{N+1}=n_{N+1}$ for any polygon, for each new loop we always add one fewer new vertexes than the number of new edges, so the quantity $v-n-1$ will always decrease by 1 for each new loop added. Since any diagram with any number of loops can be constructed by starting with a single loop and adding loops one at a time, this proves the result 5 in general.

Note that we count only loops that don't enclose other loops. In the black diagram, the path that encloses the two inner loops is *not* counted as a separate loop.

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