MUON DECAY: FEYNMAN AMPLITUDE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Section 7.2.2.

Another example of a decay rate calculation is provided by L&P in their section 7.2.2, where they consider the muon decay into an electron, a muon neutrino and an electron antineutrino:

$$\mu^{-}(p)\rightarrow e^{-}\left(p'\right)+\hat{\nu}_{e}\left(k'\right)+\nu_{\mu}(k) \tag{1}$$

In the first part of this section, they calculate the Feynman amplitude. For the most part, their derivation is clear enough, but there are a few points that need some extra explanation. The interaction Lagrangian is

$$\mathscr{L}_{\text{int}}=\frac{G_{F}}{\sqrt{2}}\overline{\psi}_{(e)}\gamma^{\lambda}(1-\gamma_{5})\psi_{(\nu_{e})}\overline{\psi}_{(\nu_{\mu})}\gamma_{\lambda}(1-\gamma_{5})\psi_{(\mu)} \tag{2}$$

where G_F is the Fermi constant. Like all Lagrangians, this one is conjured to fit the reaction and doesn't appear to be derived from anywhere.

The interaction Hamiltonian is just the negative of this:

$$\mathscr{H}_{\text{int}}=-\frac{G_{F}}{\sqrt{2}}\overline{\psi}_{(e)}\gamma^{\lambda}(1-\gamma_{5})\psi_{(\nu_{e})}\overline{\psi}_{(\nu_{\mu})}\gamma_{\lambda}(1-\gamma_{5})\psi_{(\mu)} \tag{3}$$

In this case, all particles are fermions, so to pick out the terms that don't give zero in the Feynman amplitude, we refer back to the Fourier decomposition of fermion fields

$$\psi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + \hat{f}_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{ip\cdot x} \right) \tag{4}$$

$$= \psi_+(x) + \psi_-(x) \tag{5}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \left(f_s^\dagger(\mathbf{p}) \overline{u}_s(\mathbf{p}) e^{ip\cdot x} + \hat{f}_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) e^{-ip\cdot x} \right) \tag{6}$$

$$= \overline{\psi}_-(x) + \overline{\psi}_+(x) \tag{7}$$

In the decay, we annihilate the muon and create the 3 outgoing particles. Therefore, we need $\psi_{(\mu)+}$, $\overline{\psi}_{(e)-}$, $\psi_{(\nu_e)-}$ and $\overline{\psi}_{(\nu_\mu)-}$. The initial state is $|\mu(p)\rangle$ and the final state is $\langle e^-(p')\hat{\nu}_e(k')\nu_\mu(k)|$. Applying the field operators using our earlier rules and separating out the Feynman amplitude we get

$$\mathscr{M}_{fi} = \frac{G_F}{\sqrt{2}} \overline{u}_{(e)}(p')\gamma^\lambda(1-\gamma_5)v_{(\nu_e)}(k')\overline{u}_{(\nu_\mu)}(k)\gamma_\lambda(1-\gamma_5)u_{(\mu)}(p) \tag{8}$$

To determine the Feynman amplitude, we need to sum over the spin states, which gives us L&P's equation 7.40:

There are implied sums over λ and ρ.

$$\begin{aligned}\overline{|\mathscr{M}|^2} = \frac{G_F^2}{4} \sum_{\text{spin}} & \left[\overline{u}_{(e)}(p')\gamma^\lambda(1-\gamma_5)v_{(\nu_e)}(k')\right] \\ & \times \left[\overline{u}_{(\nu_\mu)}(k)\gamma_\lambda(1-\gamma_5)u_{(\mu)}(p)\right] \\ & \times \left[\overline{u}_{(e)}(p')\gamma^\rho(1-\gamma_5)v_{(\nu_e)}(k')\right]^\dagger \\ & \times \left[\overline{u}_{(\nu_\mu)}(k)\gamma_\rho(1-\gamma_5)u_{(\mu)}(p)\right]^\dagger \end{aligned} \tag{9}$$

To work out this term, we refer back to the formulas for summing over spins that we used earlier. As before, we note that each quantity in square brackets is just a number, as it is the product of a 1×4 row vector, a 4×4 matrix and a 4×1 column vector. Thus the Hermitian conjugate operation is the same as the complex conjugate. Consider the first and third brackets in 9. It has the form

$$\sum_{\text{spin}} \left[\overline{u}_{(e)}(p')\gamma^\lambda(1-\gamma_5)v_{(\nu_e)}(k')\right]\left[\overline{u}_{(e)}(p')\gamma^\rho(1-\gamma_5)v_{(\nu_e)}(k')\right]^* \tag{10}$$

This can be written as

$$\sum_{\text{spin}}\left[\overline{u}_{(e)}\left(p'\right)Fv_{(\nu_e)}\left(k'\right)\right]\left[\overline{u}_{(e)}\left(p'\right)Gv_{(\nu_e)}\left(k'\right)\right]^* \tag{11}$$

where

$$F = \gamma^\lambda\left(1-\gamma_5\right) \tag{12}$$
$$G = \gamma^\rho\left(1-\gamma_5\right) \tag{13}$$

We've seen that

$$\left[\overline{u}_{(e)}\left(p'\right)Gv_{(\nu_e)}\left(k'\right)\right]^* = \overline{v}_{(\nu_e)}\left(k'\right)G^\ddagger u_{(e)}\left(p'\right) \tag{14}$$

with

$$G^\ddagger = \gamma^0 G^\dagger \gamma^0 \tag{15}$$

so 11 becomes

$$\sum_{\text{spin}}\left[\overline{u}_{(e)}\left(p'\right)Fv_{(\nu_e)}\left(k'\right)\right]\left[\overline{v}_{(\nu_e)}\left(k'\right)G^\ddagger u_{(e)}\left(p'\right)\right] \tag{16}$$

If we neglect the neutrino mass (which is known to be very small), and follow the same derivation as before we find that this sum is

$$\sum_{\text{spin}}\left[\overline{u}_{(e)}\left(p'\right)Fv_{(\nu_e)}\left(k'\right)\right]\left[\overline{v}_{(\nu_e)}\left(k'\right)G^\ddagger u_{(e)}\left(p'\right)\right] = \text{Tr}\left[\left(\not{p}'+m_e\right)F\not{k}'G^\ddagger\right] \tag{17}$$

We can work out $G^\ddagger$ as follows:

$$\begin{aligned}
G^\ddagger &= \gamma^0\left[\gamma^\rho\left(1-\gamma_5\right)\right]^\dagger\gamma^0 && (18)\\
&= \gamma^0\left[\gamma^0\gamma^\rho\gamma^0 - \gamma_5\gamma^0\gamma^\rho\gamma^0\right]\gamma^0 && (19)\\
&= \gamma^\rho + \gamma_5\gamma^\rho && (20)\\
&= \gamma^\rho - \gamma^\rho\gamma_5 && (21)\\
&= G && (22)
\end{aligned}$$

where we've used the properties of the gamma matrices.

We can apply the same reasoning to lines 2 and 4 in 9 and we get L&P's equation 7.41:

$$\begin{aligned}
\overline{|\mathscr{M}|^2} = \frac{G_F^2}{4}&\text{Tr}\left[\left(\not{p}'+m_e\right)\gamma^\lambda\left(1-\gamma_5\right)\not{k}'\gamma^\rho\left(1-\gamma_5\right)\right]\\
&\times\text{Tr}\left[\left(\not{k}\right)\gamma_\lambda\left(1-\gamma_5\right)\left(\not{p}+m_\mu\right)\gamma_\rho\left(1-\gamma_5\right)\right] && (23)
\end{aligned}$$

L&P then go through a couple of steps to arrive at equation 7.43:

$$\begin{aligned}\overline{|\mathscr{M}|^2} = G_F^2 \mathrm{Tr}\left[\left(\not{p}' + m_e\right)\gamma^\lambda \not{k}' \gamma^\rho (1-\gamma_5)\right] \\ \times \mathrm{Tr}\left[(\not{k})\gamma_\lambda \left(\not{p} + m_\mu\right)\gamma_\rho (1-\gamma_5)\right] \qquad (24)\end{aligned}$$

To proceed, L&P use some trace formulas for gamma matrices and slashed vectors in their Appendix A.2. Since the trace of the product of an odd number of gamma matrices (whether or not this product is multiplied by γ_5) vanishes, all terms involving the masses m_e and m_μ will vanish from the trace. They then apply some other formulas from A.2 to get their equation 7.44. However, I don't think this equation is quite right, since the positions of the indexes in the two square brackets aren't correct. All terms in the first bracket should have 2 upper indexes and all terms in the second bracket should have 2 lower indexes. To correct this, we apply the trace formulas. Consider the first trace in the first bracket of 24.

$$\mathrm{Tr}\left(\not{p}'\gamma^\lambda \not{k}'\gamma^\rho\right) = p'^\alpha k'^\beta \mathrm{Tr}\left(\gamma_\alpha \gamma^\lambda \gamma_\beta \gamma^\rho\right) \qquad (25)$$

We can apply formula A.24:

$$\mathrm{Tr}\left(\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho\right) = 4\left(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda}\right) \qquad (26)$$

This gives

$$\begin{aligned}\mathrm{Tr}\left(\not{p}'\gamma^\lambda \not{k}'\gamma^\rho\right) &= 4p'^\alpha k'^\beta \left(g_\alpha^{\ \lambda} g_\beta^{\ \rho} - g_{\alpha\beta}g^{\lambda\rho} + g_\alpha^{\ \rho} g^{\lambda}_{\ \beta}\right) \qquad (27)\\ &= 4\left(p'^\lambda k'^\rho - g^{\lambda\rho} p'\cdot k' + p'^\rho k'^\lambda\right) \qquad (28)\end{aligned}$$

The trace of the term in 24 involving γ_5 can be evaluated using formula A.26, which is

$$\mathrm{Tr}\left(\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho \gamma_5\right) = 4i\epsilon_{\mu\nu\lambda\rho} \qquad (29)$$

We then have

$$\begin{aligned}\mathrm{Tr}\left[\not{p}'\gamma^\lambda \not{k}'\gamma^\rho\gamma_5\right] &= p'^\alpha k'^\beta \mathrm{Tr}\left(\gamma_\alpha \gamma^\lambda \gamma_\beta \gamma^\rho \gamma_5\right) \qquad (30)\\ &= 4ip'^\alpha k'^\beta \epsilon_{\alpha\ \beta}^{\ \lambda\ \rho} \qquad (31)\\ &= 4ip'^\alpha k'^\beta g^{\lambda\delta} g^{\rho\gamma} \epsilon_{\alpha\delta\beta\gamma} \qquad (32)\end{aligned}$$

Since the tensor $\epsilon_{\alpha\delta\beta\gamma}$ is antisymmetric, we can swap the first and second indexes and the third and fourth indexes without changing its value, so we have

$$\text{Tr}\left[\not{p}'\gamma^{\lambda}\not{k}'\gamma^{\rho}\gamma_5\right]=4ip'^{\alpha}k'^{\beta}g^{\lambda\delta}g^{\rho\gamma}\epsilon_{\delta\alpha\gamma\beta} \tag{33}$$

Doing a similar calculation for the second bracket in 24 and inserting the result back into 24 we get what I think is the correct version:

It turns out that this doesn't affect the final answer obtained in L&P.

$$\overline{|\mathscr{M}|^2}=16G_F^2\left[p'^{\lambda}k'^{\rho}+p'^{\rho}k'^{\lambda}-g^{\lambda\rho}p'\cdot k'-ip'^{\alpha}k'^{\beta}g^{\lambda\delta}g^{\rho\gamma}\epsilon_{\delta\alpha\gamma\beta}\right]$$
$$\times\left[p_{\lambda}k_{\rho}+p_{\rho}k_{\lambda}-g_{\lambda\rho}p\cdot k-ig_{\lambda\theta}g_{\rho\xi}\epsilon^{\theta\nu\xi\eta}k_{\nu}p_{\eta}\right] \tag{34}$$

It is still true that the first 3 terms in each bracket are symmetric under the exchange $\lambda\leftrightarrow\rho$ while the last term in each bracket is antisymmetric.

L&P's equation 7.45 should therefore be

$$\overline{|\mathscr{M}|^2}=16G_F^2\left[\left(p'^{\lambda}k'^{\rho}+p'^{\rho}k'^{\lambda}-g^{\lambda\rho}p'\cdot k'\right)\left(p_{\lambda}k_{\rho}+p_{\rho}k_{\lambda}-g_{\lambda\rho}p\cdot k\right)\right.$$
$$\left.-p'^{\alpha}k'^{\beta}k_{\nu}p_{\eta}g_{\theta}^{\delta}g_{\xi}^{\gamma}\epsilon_{\delta\alpha\gamma\beta}\epsilon^{\theta\nu\xi\eta}\right] \tag{35}$$

The terms g_{θ}^{δ} and g_{ξ}^{γ} are just Kronecker deltas, so the last term contains the contraction

$$g_{\theta}^{\delta}g_{\xi}^{\gamma}\epsilon_{\delta\alpha\gamma\beta}\epsilon^{\theta\nu\xi\eta}=\epsilon_{\delta\alpha\gamma\beta}\epsilon^{\delta\nu\gamma\eta} \tag{36}$$

The remainder of the derivation in L&P is fairly straightforward, so we end up with the result in equation 7.50:

$$\Gamma=\frac{G_F^2}{\pi^5 m_{\mu}}\int\frac{d^3p'}{2p'_0}\int\frac{d^3k}{2k_0}\int\frac{d^3k'}{2k'_0}\delta^4\left(p-p'-k-k'\right)\left(p\cdot k'\right)\left(k\cdot p'\right) \tag{37}$$

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