

BOSON DECAY RATE: SUM OVER SPINS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Exercises 7.1 - 7.2.

In the calculation of the decay rate of a boson via the Yukawa interaction, we come across the term

$$\Sigma_{\text{spin}} = \sum_{s,s'} |\bar{u}_s(\mathbf{p}) v_{s'}(\mathbf{p}')|^2 \quad (1)$$

$$= \sum_{s,s'} [\bar{u}_s(\mathbf{p}) v_{s'}(\mathbf{p}')] [\bar{u}_s(\mathbf{p}) v_{s'}(\mathbf{p}')]^* \quad (2)$$

L&P work out the complex conjugate term in their equation 7.13, but we can work out a more general case as follows. For a general 4×4 matrix F we wish to find $[\bar{u}Fv]^*$. I've suppressed the dependence of u and v on spin and momentum to make the notation easier. First, note that since \bar{u} is a 1×4 matrix (a row vector), F is a 4×4 matrix and v is a 4×1 matrix (a column vector), the product $\bar{u}Fv$ is just a single number (or a 1×1 matrix if you prefer), so its complex conjugate is also its hermitian conjugate. Therefore

$$[\bar{u}Fv]^* = [\bar{u}Fv]^\dagger \quad (3)$$

$$= [u^\dagger \gamma_0 F v]^\dagger \quad (4)$$

$$= v^\dagger F^\dagger \gamma_0 u \quad (5)$$

$$= v^\dagger \gamma_0 \gamma_0 F^\dagger \gamma_0 u \quad (6)$$

$$= \bar{v} F^\ddagger u \quad (7)$$

where

$$F^\ddagger \equiv \gamma_0 F^\dagger \gamma_0 \quad (8)$$

and we've used the properties of γ_0 : $\gamma_0^\dagger = \gamma_0$ and $(\gamma_0)^2 = I$.

When we use this result in 2 we have

When we multiply a 4×1 matrix into a 1×4 matrix, the result is a 4×4 matrix.

$$\Sigma_{\text{spin}} = \sum_{s,s'} [(\bar{u}_s(\mathbf{p}))_\alpha (v_{s'}(\mathbf{p}'))_\alpha] [(\bar{v}_{s'}(\mathbf{p}'))_\beta (u_s(\mathbf{p}))_\beta] \quad (9)$$

$$= \left[\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) \right]_{\beta\alpha} \left[\sum_{s'} v_{s'}(\mathbf{p}') \bar{v}_{s'}(\mathbf{p}') \right]_{\alpha\beta} \quad (10)$$

where the subscripts α and β refer to the components of the matrices. Again, we can work out this term for the more general case.

All Greek subscripts are summed from 1 to

$$\sum_{s,s'} |\bar{u}_s(\mathbf{p}) F v_{s'}(\mathbf{p}')|^2 = \sum_{s,s'} [(\bar{u}_s(\mathbf{p}))_\alpha F_{\alpha\beta} (v_{s'}(\mathbf{p}'))_\beta] [(\bar{v}_{s'}(\mathbf{p}'))_\delta F_{\delta\epsilon}^\dagger (u_s(\mathbf{p}))_\epsilon] \quad (11)$$

$$= \left[\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) \right]_{\epsilon\alpha} F_{\alpha\beta} \left[\sum_{s'} v_{s'}(\mathbf{p}') \bar{v}_{s'}(\mathbf{p}') \right]_{\beta\delta} F_{\delta\epsilon}^\dagger \quad (12)$$

We can now use the results

$$\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) = \not{p} + m \quad (13)$$

$$\sum_s v_s(\mathbf{p}) \bar{v}_s(\mathbf{p}) = \not{p} - m \quad (14)$$

We get

$$\sum_{s,s'} |\bar{u}_s(\mathbf{p}) F v_{s'}(\mathbf{p}')|^2 = (\not{p} + m)_{\epsilon\alpha} F_{\alpha\beta} (\not{p}' - m)_{\beta\delta} F_{\delta\epsilon}^\dagger \quad (15)$$

Note that summing over the Greek subscripts means that the four terms are just a matrix product, with one extra sum over the first and last indexes. That is

$$(\not{p} + m)_{\epsilon\alpha} F_{\alpha\beta} (\not{p}' - m)_{\beta\delta} F_{\delta\epsilon}^\dagger = \sum_\epsilon [(\not{p} + m) F (\not{p}' - m) F^\dagger]_{\epsilon\epsilon} \quad (16)$$

$$= \text{Tr} [(\not{p} + m) F (\not{p}' - m) F^\dagger] \quad (17)$$

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