BOSON DECAY RATE: SUM OVER SPINS

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Post date: 10 Aug 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Exercises 7.1 - 7.2.

In the calculation of the decay rate of a boson via the Yukawa interaction, we come across the term

$$\Sigma_{\text{spin}} = \sum_{s,s'} \left|\overline{u}_s(\mathbf{p})\, v_{s'}(\mathbf{p}')\right|^2 \tag{1}$$

$$= \sum_{s,s'} \left[\overline{u}_s(\mathbf{p})\, v_{s'}(\mathbf{p}')\right]\left[\overline{u}_s(\mathbf{p})\, v_{s'}(\mathbf{p}')\right]^* \tag{2}$$

L&P work out the complex conjugate term in their equation 7.13, but we can work out a more general case as follows. For a general 4×4 matrix F we wish to find $[\overline{u}Fv]^*$. I've suppressed the dependence of u and v on spin and momentum to make the notaton easier. First, note that since $\overline{u}$ is a 1×4 matrix (a row vector), F is a 4×4 matrix and v is a 4×1 matrix (a column vector), the product $\overline{u}Fv$ is just a single number (or a 1×1 matrix if you prefer), so its complex conjugate is also its hermitian conjugate. Therefore

$$[\overline{u}Fv]^* = [\overline{u}Fv]^\dagger \tag{3}$$

$$= \left[u^\dagger \gamma_0 F v\right]^\dagger \tag{4}$$

$$= v^\dagger F^\dagger \gamma_0 u \tag{5}$$

$$= v^\dagger \gamma_0 \gamma_0 F^\dagger \gamma_0 u \tag{6}$$

$$= \overline{v} F^\ddagger u \tag{7}$$

where

$$F^\ddagger \equiv \gamma_0 F^\dagger \gamma_0 \tag{8}$$

and we've used the properties of γ_0: $\gamma_0^\dagger = \gamma_0$ and $(\gamma_0)^2 = I$.

When we use this result in 2 we have

When we multiply a 4×1 matrix into a 1×4 matrix, the result is a 4×4 matrix.

$$\Sigma_{\text{spin}} = \sum_{s,s'} \left[\left(\overline{u}_s (\mathbf{p}) \right)_\alpha \left(v_{s'} (\mathbf{p}') \right)_\alpha \right] \left[\left(\overline{v}_{s'} (\mathbf{p}') \right)_\beta \left(u_s (\mathbf{p}) \right)_\beta \right] \tag{9}$$

$$= \left[\sum_s u_s (\mathbf{p}) \overline{u}_s (\mathbf{p}) \right]_{\beta\alpha} \left[\sum_{s'} v_{s'} (\mathbf{p}') \overline{v}_{s'} (\mathbf{p}') \right]_{\alpha\beta} \tag{10}$$

where the subscripts α and β refer to the components of the matrices. Again, we can work out this term for the more general case.

All Greek subscripts are summed from 1 to 4.

$$\sum_{s,s'} \left| \overline{u}_s (\mathbf{p}) F v_{s'} (\mathbf{p}') \right|^2 = \sum_{s,s'} \left[\left(\overline{u}_s (\mathbf{p}) \right)_\alpha F_{\alpha\beta} \left(v_{s'} (\mathbf{p}') \right)_\beta \right] \left[\left(\overline{v}_{s'} (\mathbf{p}') \right)_\delta F^{\ddagger}_{\delta\epsilon} \left(u_s (\mathbf{p}) \right)_\epsilon \right] \tag{11}$$

$$= \left[\sum_s u_s (\mathbf{p}) \overline{u}_s (\mathbf{p}) \right]_{\epsilon\alpha} F_{\alpha\beta} \left[\sum_{s'} v_{s'} (\mathbf{p}') \overline{v}_{s'} (\mathbf{p}') \right]_{\beta\delta} F^{\ddagger}_{\delta\epsilon} \tag{12}$$

We can now use the results

$$\sum_s u_s (\mathbf{p}) \overline{u}_s (\mathbf{p}) = \not{p} + m \tag{13}$$

$$\sum_s v_s (\mathbf{p}) \overline{v}_s (\mathbf{p}) = \not{p} - m \tag{14}$$

We get

$$\sum_{s,s'} \left| \overline{u}_s (\mathbf{p}) F v_{s'} (\mathbf{p}') \right|^2 = \left(\not{p} + m \right)_{\epsilon\alpha} F_{\alpha\beta} \left(\not{p}' - m \right)_{\beta\delta} F^{\ddagger}_{\delta\epsilon} \tag{15}$$

Note that summing over the Greek subscripts means that the four terms are just a matrix product, with one extra sum over the first and last indexes. That is

$$\left(\not{p} + m \right)_{\epsilon\alpha} F_{\alpha\beta} \left(\not{p}' - m \right)_{\beta\delta} F^{\ddagger}_{\delta\epsilon} = \sum_\epsilon \left[\left(\not{p} + m \right) F \left(\not{p}' - m \right) F^{\ddagger} \right]_{\epsilon\epsilon} \tag{16}$$

$$= \text{Tr} \left[\left(\not{p} + m \right) F \left(\not{p}' - m \right) F^{\ddagger} \right] \tag{17}$$

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