MUON DECAY: EFFECT OF ELECTRON MASS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 17 Aug 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Exercise 7.5.

Here we'll continue our study of the muon decay rate for the process

$$\mu^{-}(p)\rightarrow e^{-}\left(p'\right)+\hat{\nu}_{e}\left(k'\right)+\nu_{\mu}(k) \tag{1}$$

The result for the given in L&P's equation 7.75 neglects the electron mass:

$$\Gamma=\frac{G_{F}^{2}m_{\mu}^{5}}{192\pi^{3}} \tag{2}$$

The problem is to determine the lowest order correction to this value that can be found by including the electron mass. The result 2 is found by integrating the general expression for the lowest order term in the Wick expansion, which is given in L&P's equation 7.72:

$$\begin{aligned}\frac{d\Gamma}{dE_{e}}=\frac{G_{F}^{2}}{12\pi^{3}}\sqrt{E_{e}^{2}-m_{e}^{2}}\left[\left(m_{\mu}^{2}+m_{e}^{2}-2m_{\mu}E_{e}\right)E_{e}\right.\\ \left.+2\left(m_{\mu}-E_{e}\right)\left(m_{\mu}E_{e}-m_{e}^{2}\right)\right]\end{aligned} \tag{3}$$

The first thought might be that we need to expand this expression in powers of m_e and then just integrate the lowest order non-zero power of m_e to get the required correction. However, the problem is that we need to know the limits of integration. If $m_e=0$, L&P show that the lower limit for E_e is zero, since the electron could be produced at rest, with the two neutrinos carrying off all the momentum. (This ignores the complication that a particle with zero rest mass must always travel at the speed of light, but never mind.) The upper limit is obtained when the two neutrinos both go off in the same direction, so that the electron must have a momentum that is equal and opposite to the sum of the two neutrino momenta. Conservation of energy leads to a maximum value of E_e of $m_\mu/2$.

When we assume a non-zero value for m_e, however, neither of these limits is correct. The lower limit, when the electron is at rest, is now m_e. The upper limit still occurs when the two neutrinos both go off in the same direction, so that the electron's momentum must balance that of the neutrinos. Since we're assuming that the neutrinos are massless and the muon decays at rest, the conservation of energy condition is

$$E_e + k_0 + k_0' = m_\mu \tag{4}$$

The maximum momentum for the electron has a magnitude $|\mathbf{p}'|$ which is the sum of the magnitudes of the two neutrino momenta:

$$\left|\mathbf{p}'\right| = |\mathbf{k}| + \left|\mathbf{k}'\right| \tag{5}$$

However, because the neutrinos are massless, the energy of each is just the magnitude of the 3-momentum, so we have

$$\left|\mathbf{p}'\right| = k_0 + k_0' \tag{6}$$

The energy of the electron is given by

$$\begin{aligned} E_e &= \sqrt{(\mathbf{p}')^2 + m_e^2} &(7)\\ &= \sqrt{\left(k_0 + k_0'\right)^2 + m_e^2} &(8)\\ &= \sqrt{(m_\mu - E_e)^2 + m_e^2} &(9) \end{aligned}$$

We can solve this to obtain

$$E_e = \frac{m_e^2 + m_\mu^2}{2m_\mu} \tag{10}$$

which is now the upper limit of integration.

The presence of m_e in the limit of integration complicates things since we can't just expand the integrand to the lowest power of m_e and integrate from there. There are two ways of solving the problem. We can integrate 3 exactly (the integral does have a closed form solution, although it's quite messy) and then plug in the limits and expand the result in a series. Or we can expand the integrand out to the lowest order in m_e and integrate the entire expansion.

I used Maple to do the integrals and series expansions. If we do the integral exactly, plug in the limits and expand the result, we get

$$\Gamma = \int_{m_e}^{\left(m_e^2+m_\mu^2\right)/2m_\mu} \frac{d\Gamma}{dE_e} dE_e \tag{11}$$

$$= \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - 8\frac{m_e^2}{m_\mu^2} + \dots\right) \tag{12}$$

If we expand 3 out to the first non-zero power of m_e we get

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2}{\pi^3}\left[\frac{(3m_\mu - 4E_e)\,E_e^2 m_\mu}{12} + \frac{2E_e^2 - m_\mu^2}{8} m_e^2 + \dots\right] \tag{13}$$

Integrating this entire expression between the above limits and then taking the leading terms in the result also gives 12. Probably the second method would be easier to do by hand, although neither of them is particularly nice.

Using the values $m_e = 0.511$ MeV and $m_\mu = 106$ MeV, the lowest order correction amounts to

$$8\frac{m_e^2}{m_\mu^2} = 0.000186 \tag{14}$$

so the correction is quite small.