CHARGED PION DECAY

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Exercise 7.6.

The decay of a charged pion (a boson) to a lepton (either a muon or electron) and a neutrino is given by the processes

Actually, the decay products are an antimuon in the first case and a positron (antielectron) in the second.

$$\pi^+ \to \mu^+ + \nu_\mu \tag{1}$$

$$\pi^+ \to e^+ + \nu_e \tag{2}$$

The interaction Lagrangian (again, god-given) is

$$\mathscr{L}_{\text{int}} = G_F f_\pi (\partial_\lambda \phi) \overline{\psi}_\ell \gamma^\lambda (1-\gamma_5) \psi_\nu + \text{h.c.} \tag{3}$$

where 'h.c.' is the Hermitian conjugate. The parameters G_F and f_π are constants. ϕ is the scalar field representing the pion, ψ_ℓ is the lepton (muon or electron) field and ψ_ν is the neutrino field. In what follows, k is the four-momentum of the pion, p of the lepton and q of the neutrino. We'll assume the neutrino has zero rest mass, and the rest mass of the lepton is m_ℓ where ℓ can be either μ or e.

In order to get the correct creation and annihilation operators, we actually need to use the 'h.c.' term in the Lagrangian, which is

$$\text{h.c.} = G_F f_\pi (\partial_\lambda \phi) \psi_\nu^\dagger (1-\gamma_5) \gamma^0 \gamma^\lambda \gamma^0 \gamma^0 \psi_\ell \tag{4}$$

$$= G_F f_\pi (\partial_\lambda \phi) \overline{\psi}_\nu (1+\gamma_5) \gamma^\lambda \psi_\ell \tag{5}$$

$$= G_F f_\pi (\partial_\lambda \phi) \gamma^\lambda (1-\gamma_5) \psi_\ell \tag{6}$$

From the Fourier decomposition of the scalar field we have

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \left(a(k) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x} \right) \tag{7}$$

and the term that annihilates the pion is

$$\phi_{+}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^{3}k}{\sqrt{2E_{k}}}a(k)\,e^{-ik\cdot x} \tag{8}$$

The derivative $\partial_\lambda\phi$ in 3 thus brings down a factor of $-ik_\lambda$ and since we are going to be taking the square modulus when calculating the Feynman amplitude, is equivalent to multiplying by just k_λ. By using the usual prescriptions for constructing the matrix element we get the lowest order term in the Feynman amplitude:

$$\mathscr{M}_{fi}=G_{F}f_{\pi}k_{\lambda}\overline{u}_{\nu}\gamma^{\lambda}(1-\gamma_{5})\,v_{\ell} \tag{9}$$

To get the full Feynman amplitude, we follow a similar procedure that of the earlier post, and sum over final spins. This gives

$$|\mathscr{M}|^{2}=(G_{F}f_{\pi})^{2}\sum_{\text{spin}}\left[k_{\lambda}\overline{u}_{\nu}\gamma^{\lambda}(1-\gamma_{5})\,v_{\ell}\right]\left[k_{\lambda}\overline{u}_{\nu}\gamma^{\lambda}(1-\gamma_{5})\,v_{\ell}\right]^{\dagger} \tag{10}$$

Before evaluating this, we note that we can use $\not{k}\equiv k_\lambda\gamma^\lambda$ to write the first bracket as

$$k_{\lambda}\overline{u}_{\nu}\gamma^{\lambda}(1-\gamma_{5})\,v_{\ell}=\overline{u}_{\nu}\not{k}(1-\gamma_{5})\,v_{\ell} \tag{11}$$

From four-momentum conservation, we have

$$k=p+q \tag{12}$$

For fermions we also have the relations

$$\left(\not{p}-m\right)u_{s}(\mathbf{p})=0 \tag{13}$$
$$\left(\not{p}+m\right)v_{s}(\mathbf{p})=0 \tag{14}$$

$$\overline{u}_{s}(\mathbf{p})\left(\not{p}-m\right)=0 \tag{15}$$
$$\overline{v}_{s}(\mathbf{p})\left(\not{p}+m\right)=0 \tag{16}$$

We can use these and the anticommutation properties of the gamma matrices to write 11 as

$$\overline{u}_{\nu}\not{k}(1-\gamma_{5})\,v_{\ell}=\overline{u}_{\nu}\left(\not{p}+\not{q}\right)(1-\gamma_{5})\,v_{\ell} \tag{17}$$
$$=\overline{u}_{\nu}(1+\gamma_{5})\not{p}v_{\ell}+\overline{u}_{\nu}\not{q}(1-\gamma_{5})\,v_{\ell} \tag{18}$$

Since we're taking the neutrino mass to be zero,

$$\overline{u}_{\nu}\not{q}=0 \tag{19}$$

and for the lepton

$$\not{p} v_\ell = -m_\ell v_\ell \tag{20}$$

so we have

$$\overline{u}_\nu \not{k} (1-\gamma_5) v_\ell = -m_\ell \overline{u}_\nu (1+\gamma_5) v_\ell \tag{21}$$

Inserting this into 10 we have

$$|\mathscr{M}|^2 = (G_F f_\pi)^2 m_\ell^2 \sum_{\text{spin}} \left[\overline{u}_\nu(q)(1+\gamma_5) v_\ell(p)\right]\left[\overline{u}_\nu(q)(1+\gamma_5) v_\ell(p)\right]^\dagger \tag{22}$$

where we've inserted the momentum dependencies explicitly.

We can now use the earlier result

$$\sum_{s,s'} \left|\overline{u}_s(\mathbf{p}) F v_{s'}(\mathbf{p}')\right|^2 = \text{Tr}\left[\left(\not{p}+m\right) F \left(\not{p}'-m\right) F^\ddagger\right] \tag{23}$$

where

$$F^\ddagger \equiv \gamma_0 F^\dagger \gamma_0 \tag{24}$$

to write this as

$$|\mathscr{M}|^2 = (G_F f_\pi)^2 m_\ell^2 \text{Tr}\left[\not{q}(1+\gamma_5)\left(\not{p}+m_\ell\right)\gamma^0(1+\gamma_5)\gamma^0\right] \tag{25}$$

$$= (G_F f_\pi)^2 m_\ell^2 \text{Tr}\left[\not{q}(1+\gamma_5)\left(\not{p}+m_\ell\right)(1-\gamma_5)\right] \tag{26}$$

where we've used $\left\{\gamma^0, \gamma_5\right\} = 0$ and $\left(\gamma^0\right)^2 = I$. We now need some trace formulas from the appendix in L&P. We have

$$\text{Tr}\left[\gamma_\mu \gamma_\nu \gamma_5\right] = 0 \tag{27}$$

so using the cyclic property of the trace

$$\text{Tr}\left[\not{q}\gamma_5\not{p}\right] = \text{Tr}\left[\not{q}\not{p}\gamma_5\right] = 0 \tag{28}$$

Therefore we have

$$|\mathscr{M}|^2 = (G_F f_\pi)^2 m_\ell^2 \text{Tr}\left[\not{q}\not{p} - \not{q}\gamma_5\not{p}\gamma_5\right] \tag{29}$$

$$= (G_F f_\pi)^2 m_\ell^2 \text{Tr}\left[\not{q}\not{p} + \not{q}\not{p}(\gamma_5)^2\right] \tag{30}$$

$$= 2(G_F f_\pi)^2 m_\ell^2 \text{Tr}\left[\not{q}\not{p}\right] \tag{31}$$

Using L&P's equation A.19:

$$\text{Tr}\left(\not{p}\not{q}\right) = 4p \cdot q \tag{32}$$

we have

$$|\mathscr{M}|^2 = 8\left(G_F f_\pi\right)^2 m_\ell^2 p \cdot q \tag{33}$$

We can get the four-vector product from energy conservation, since

$$k^2 = (p+q)^2 = p^2 + q^2 + 2p \cdot q \tag{34}$$

For the pion $k^2 = m_\pi^2$, for the lepton $p^2 = m_\ell^2$ and for the neutrino $q^2 = 0$ so

$$p \cdot q = \frac{m_\pi^2 - m_\ell^2}{2} \tag{35}$$

The final result for the Feynman amplitude is thus

$$|\mathscr{M}|^2 = 4\left(G_F f_\pi\right)^2 m_\ell^2 \left(m_\pi^2 - m_\ell^2\right) \tag{36}$$

To get the actual decay rate Γ we now need to do the integral obtained from L&P's equation 7.8. If we assume the pion decays at rest, then

$$\Gamma = \frac{1}{2m_\pi} \int \frac{d^3p}{(2\pi)^3 2p_0} \int \frac{d^3q}{(2\pi)^3 2q_0} (2\pi)^4 \delta^4 (k - p - q) |\mathscr{M}|^2 \tag{37}$$

We see from 36 that $|\mathscr{M}|^2$ is a constant, so we can ignore it when calculating the integral. We can write

$$\Gamma = \frac{|\mathscr{M}|^2}{2m_\pi} \rho \tag{38}$$

where

$$\rho \equiv \int \frac{d^3p}{(2\pi)^3 2p_0} \int \frac{d^3q}{(2\pi)^3 2q_0} (2\pi)^4 \delta^4 (k - p - q) \tag{39}$$

As the integrand doesn't depend on the spatial components of q, we can integrate these to give

$$\rho = \frac{1}{(2\pi)^2} \int \frac{d^3p}{4p_0 q_0} \delta (k_0 - p_0 - q_0) \tag{40}$$

Since we're assuming the decaying pion is at rest, the momentum of the lepton must be equal and opposite to the momentum of the neutrino. If we call the magnitude of this momentum $\mathfrak{p}$ then

$$p_0 = \sqrt{m_\ell^2 + \mathfrak{p}^2} \tag{41}$$
$$q_0 = \mathfrak{p} \tag{42}$$
$$k_0 = m_\pi \tag{43}$$

We now need to transform the delta function. To do this we use the formula (also L&P equation 3.15):

$$\delta(f(\mathfrak{p})) = \frac{\delta(\mathfrak{p} - \mathfrak{p}_0)}{|f'(\mathfrak{p}_0)|} \tag{44}$$

where $f(\mathfrak{p}_0) = 0$.

Here we have

$$f(\mathfrak{p}) = k_0 - p_0 - q_0 \tag{45}$$
$$= m_\pi - \sqrt{m_\ell^2 + \mathfrak{p}^2} - \mathfrak{p} \tag{46}$$
$$\mathfrak{p}_0 = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \tag{47}$$
$$f'(\mathfrak{p}) = -1 - \frac{\mathfrak{p}}{\sqrt{m_\ell^2 + \mathfrak{p}^2}} \tag{48}$$

Inserting 47 into 48 and simplifying, we get

$$\left|f'(\mathfrak{p}_0)\right| = \frac{2m_\pi^2}{m_\pi^2 + m_\ell^2} \tag{49}$$

Thus the delta function in 40 becomes

$$\delta(k_0 - p_0 - q_0) = \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}\delta\left(\mathfrak{p} - \frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right) \tag{50}$$

To integrate 40, the angular integral contributes a factor of 4π since the integrand doesn't depend on direction, so we're left with

$$\rho = \frac{1}{4\pi}\frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}\int \frac{\mathfrak{p}^2 d\mathfrak{p}}{\mathfrak{p}\sqrt{m_\ell^2 + \mathfrak{p}^2}}\delta\left(\mathfrak{p} - \frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right) \tag{51}$$
$$= \frac{1}{4\pi}\frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}\int \frac{\mathfrak{p} d\mathfrak{p}}{\sqrt{m_\ell^2 + \mathfrak{p}^2}}\delta\left(\mathfrak{p} - \frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right) \tag{52}$$

To integrate this, we just set

$$\mathfrak{p} = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \tag{53}$$

in the integrand and do a bit of algebra to simplify the result, which is

$$\rho = \frac{1}{8\pi}\frac{m_\pi^2 - m_\ell^2}{m_\pi^2} \tag{54}$$

Inserting this into 38 and using 36 we get for the final result

$$\Gamma = \frac{1}{4\pi}\left(G_F f_\pi\right)^2 \frac{m_\ell^2}{m_\pi^3}\left(m_\pi^2 - m_\ell^2\right)^2 \tag{55}$$

The ratio of the decay rates to the electron and muon is therefore

$$\frac{\Gamma_e}{\Gamma_\mu} = \frac{m_e^2\left(m_\pi^2 - m_e^2\right)^2}{m_\mu^2\left(m_\pi^2 - m_\mu^2\right)^2} \tag{56}$$