SCATTERING CROSS-SECTION APPLIED TO BOSON DECAY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 26 Aug 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7, Exercise 7.7.

In their section 7.3, L&P derive a general formula for 2-particle scattering, given in their equation 7.85:

$$\sigma=\frac{1}{\rho vV}\frac{1}{4E_1E_2}\int\left(\prod_f\frac{d^3p_f}{(2\pi)^3\,2E_f}\right)(2\pi)^4\,\delta^4\left(\sum_i p_i-\sum_f p_f\right)\left|\mathscr{M}_{fi}\right|^2 \tag{1}$$

where ρ is the density in particles per unit volume of the incoming beam, v is the velocity of this beam, E_1 and E_2 are the energies of the two initial particles, and the subscripts i and f refer to initial and final particles. The Feynman amplitude is $\mathscr{M}_{fi}$ and is the only part of the formula that depends on the specific interaction. The incident flux is normalized to one particle per unit volume so

$$\rho=\frac{1}{V} \tag{2}$$

L&P then state that this formula applies strictly only to the case where the target particle is at rest, and v refers to the velocity of the incoming particle which hits the target. In practice, both particles could be moving and L&P give a curious formula for the relative velocity:

$$v_{\text{rel}}=\frac{\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}}{E_1E_2} \tag{3}$$

If we use the usual formula for the four-momentum:

$$p_i=\gamma_i\,(m_i,m_i\mathbf{v}_i) \tag{4}$$

where γ_i is the usual relativistic factor

$$\gamma_i=\frac{1}{\sqrt{1-v_i^2}} \tag{5}$$

then this formula becomes

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \gamma_1^2 \gamma_2^2 m_1^2 m_2^2 \left[(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \left(1 - v_1^2\right)(1 - v_2)^2 \right] \quad (6)$$

$$= \gamma_1^2 \gamma_2^2 m_1^2 m_2^2 \left[-2\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + v_1^2 + v_2^2 - v_1^2 v_2^2 \right] \quad (7)$$

$$= E_1^2 E_2^2 \left((\mathbf{v}_1 - \mathbf{v}_2)^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - v_1^2 v_2^2 \right) \quad (8)$$

If the velocities are small (much less that the speed of light, where $v = 1$), then the last two terms can be neglected and 3 becomes

$$v_{\text{rel}} = |\mathbf{v}_1 - \mathbf{v}_2| \quad (9)$$

This formula is valid for small velocities, since for relativistic speeds, we need to use the velocity addition formula to work out relative velocities.

Putting this together, we arrive at L&P's equation 7.88:

$$\sigma = \frac{1}{v_{\text{rel}}} \frac{1}{4E_1 E_2} \int \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f \right) |\mathscr{M}_{fi}|^2 \quad (10)$$

L&P then go through the example of 2-to-2 scattering (two incident particles and two final particles) in both the centre of mass and lab frames.

The problem here is to apply the scattering formula to the case of the decay of a single particle at rest into two final particles. The initial particle has mass M and the decay particles have masses m_1 and m_2. To apply a scattering formula to the decay, we can assume that the initial state consists of a single particle being hit by a massless particle with vanishingly small momentum. In this case, there is only one initial energy in 1. Also, as the incoming 'particle' is massless, its velocity is the speed of light so $v_{\text{rel}} = 1$ and we have

$$\sigma = \frac{1}{2M} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4 (M - p_1 - p_2) |\mathscr{M}_{fi}|^2 \quad (11)$$

In this case, the lab and centre of mass frames are the same, so we can use either in the calculation. We'll follow L&P's centre of mass example. The outgoing particles must have equal and opposite momentum so

$$\mathbf{p}_1 = -\mathbf{p}_2 \quad (12)$$

Because the masses of the final particles may be different, $E_1 \neq E_2$ in general. We can then do the integral over $d^3 p_2$ in 11 so we have

$$\sigma = \frac{1}{32\pi^2 M}\int \frac{d^3p_1}{E_1 E_2}\delta\left(M - E_1 - E_2\right)\left|\mathscr{M}_{fi}\right|^2 \tag{13}$$

Since the magnitudes of the two final momenta are equal, we can call them just $\mathfrak{p}$. Then we can use the formula

$$E_1^2 = \mathfrak{p}^2 + m_1^2 \tag{14}$$

to get

$$E_1 dE_1 = \mathfrak{p}\; d\mathfrak{p} \tag{15}$$

so we have

$$d^3p_1 = \mathfrak{p}^2 d\mathfrak{p}\; d\Omega = \mathfrak{p} E_1 dE_1 d\Omega \tag{16}$$

We therefore have

$$\sigma = \frac{1}{32\pi^2 M}\int \frac{\mathfrak{p}\; dE_1 d\Omega}{E_2}\delta\left(M - E_1 - E_2\right)\left|\mathscr{M}_{fi}\right|^2 \tag{17}$$

Since

$$E_2^2 = \mathfrak{p}^2 + m_2^2 \tag{18}$$

the other final energy E_2 is given by

$$E_2 = \sqrt{\mathfrak{p}^2 + m_2^2} = \sqrt{E_1^2 - m_1^2 + m_2^2} \tag{19}$$

The delta function in 17 is therefore

$$\delta\left(M - E_1 - E_2\right) = \delta\left(M - E_1 - \sqrt{E_1^2 - m_1^2 + m_2^2}\right) \tag{20}$$

We can transform the delta function using the formula

$$\delta\left(f(z)\right) = \frac{\delta\left(z - z_0\right)}{\left|df/dz\right|_{z=z_0}} \tag{21}$$

Here the function f is

$$f\left(E_1\right) = M - E_1 - \sqrt{E_1^2 - m_1^2 + m_2^2} \tag{22}$$

which has a zero at

$$E_{10} = \frac{M^2 + m_1^2 - m_2^2}{2M} \equiv \omega_1 \tag{23}$$

At this value, the corresponding value of E_2 is, from 19

Note that $\omega_1 + \omega_2 = M$, so energy is conserved.

$$E_{20} = \frac{M^2 - m_1^2 + m_2^2}{2M} \equiv \omega_2 \tag{24}$$

The derivative is

$$\frac{df}{dE_1} = -1 - \frac{E_1}{\sqrt{E_1^2 - m_1^2 + m_2^2}} \tag{25}$$

which has the value at $E_1 = E_{10}$ of

$$\left|\frac{df}{dE_1}\right|_{E_{10}} = \frac{2M^2}{M^2 - m_1^2 + m_2^2} = \frac{M}{\omega_2} \tag{26}$$

Thus the delta function transforms according to

$$\delta\left(M - E_1 - \sqrt{E_1^2 - m_1^2 + m_2^2}\right) = \delta\left(E_1 - \omega_1\right)\frac{\omega_2}{M} \tag{27}$$

Returning to 17, when we integrate over E_1, E_2 becomes ω_2 so we have

$$\sigma = \frac{1}{32\pi^2 M^2}\int d\Omega\, \mathfrak{p}\left|\mathscr{M}_{fi}\right|^2 \tag{28}$$

where both $\mathfrak{p}$ and $\left|\mathscr{M}_{fi}\right|^2$ are evaluated at $E_1 = \omega_1$, and the remaining integral is over the angular coordinates. We can write this as a differential cross section as:

$$\frac{d\sigma}{d\Omega} = \frac{1}{32\pi^2 M^2}\mathfrak{p}\left|\mathscr{M}_{fi}\right|^2 \tag{29}$$

To complete the calculation, we need to express $\mathfrak{p}$ in terms of the masses. We know that

$$E_1^2 = \mathfrak{p}^2 + m_1^2 \tag{30}$$

and from 23, $E_1 = \omega_1$ so we have

$$\mathfrak{p}^2 = \left[\frac{M^2 + m_1^2 - m_2^2}{2M}\right]^2 - m_1^2 \tag{31}$$

$$= \frac{1}{4M^2}\left[M^4 + m_1^4 + m_2^4 + 2M^2 m_1^2 - 2M^2 m_2^2 - 2m_1^2 m_2^2 - 4M^2 m_1^2\right] \tag{32}$$

$$= \frac{1}{4M^2}\left[M^4 + m_1^4 + m_2^4 - 2M^2 m_1^2 - 2M^2 m_2^2 - 2m_1^2 m_2^2\right] \tag{33}$$

This can be factored into

The easiest way to see this is just to multiply it out and compare with 33.

$$\mathfrak{p}^2 = \frac{1}{4M^2}\left[M^2-(m_1+m_2)^2\right]\left[M^2-(m_1-m_2)^2\right] \tag{34}$$

$$\mathfrak{p} = \frac{M}{2}\sqrt{\left[1-\left(\frac{m_1+m_2}{M}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{M}\right)^2\right]} \tag{35}$$

Plugging this into 29 we get

$$\frac{d\sigma}{d\Omega} = \frac{\left|\mathscr{M}_{fi}\right|^2}{64\pi^2 M}\sqrt{\left[1-\left(\frac{m_1+m_2}{M}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{M}\right)^2\right]} \tag{36}$$

To compare this formula with that for the decay of a boson into two fermions with the same mass, we can use the Feynman amplitude

$$\left|\mathscr{M}_{fi}\right|^2 = 2\left(M^2-4m^2\right)h^2 \tag{37}$$

Integrating 36 over the angular coordinates $d\Omega$ just multiplies the result by 4π since nothing depends on the angles, so setting $m_1 = m_2$ we get (writing Γ instead of σ for the decay rate):

$$\Gamma = \int \frac{d\sigma}{d\Omega} d\Omega \tag{38}$$

$$= \frac{4\pi}{64\pi^2 M} \times 2\left(M^2-4m^2\right)h^2\sqrt{1-\left(\frac{2m}{M}\right)^2} \tag{39}$$

$$= \frac{Mh^2}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{3/2} \tag{40}$$

which agrees with L&P's equation 7.34.