PROPAGATOR FOR THE PROCA LAGRANGIAN

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.1.

As a first step in the quantization of the electromagnetic field, we look at attempts to calculate the propagator of the field. We'll look at the Proca Lagrangian, which is similar to the standard electromagnetic Lagrangian except that is gives the particle a mass M. The Lagrangian is

$$\mathscr{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}M^{2}A^{\mu}A_{\mu}-A_{\mu}j^{\mu} \tag{1}$$

Here $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu}=\left[\begin{array}{cccc}0 & E_{x} & E_{y} & E_{z}\\ -E_{x} & 0 & -B_{z} & B_{y}\\ -E_{y} & B_{z} & 0 & -B_{x}\\ -E_{z} & -B_{y} & B_{x} & 0\end{array}\right] \tag{2}$$

This is the negative of the tensor defined in Moore's book, but the important point is that it's antisymmetric.

and j^{μ} is the 4-current. The components of $F_{\mu\nu}$ can be written in terms of the 4-potential A_{μ}as

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu} \tag{3}$$

The first step is to obtain the Euler-Lagrange equations of motion from 1. We have

$$\frac{\partial\mathcal{L}}{\partial\phi^{r}}=\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi^{r}\right)}\right) \tag{4}$$

In our case, the fields ϕ^{r} are given by the potentials A_{μ}, so we first write out 1 as

$$\mathscr{L}=-\frac{1}{4}\left(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\right)\left(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}\right)+\frac{1}{2}M^{2}A^{\mu}A_{\mu}-A_{\mu}j^{\mu} \tag{5}$$

The LHS of 4 is

$$\frac{\partial\mathcal{L}}{\partial A_{\nu}}=M^{2}A^{\nu}-j^{\nu} \tag{6}$$

If you're bothered by the fact that the term $\frac{1}{2}M^2A^\mu A_\mu$ has an upper and lower index, we can rewrite it as

$$\frac{1}{2}M^2A^\mu A_\mu = \frac{1}{2}M^2g^{\lambda\mu}A_\lambda A_\mu \tag{7}$$

so that

$$\frac{\partial}{\partial A_\nu}\left(\frac{1}{2}M^2g^{\lambda\mu}A_\lambda A_\mu\right) = \frac{1}{2}M^2g^{\lambda\mu}\left(\delta_{\lambda\nu}A_\mu + A_\lambda\delta_{\mu\nu}\right) \tag{8}$$

$$= \frac{1}{2}M^2\left(g^{\nu\mu}A_\mu + g^{\lambda\nu}A_\lambda\right) \tag{9}$$

$$= \frac{1}{2}M^2\left(A^\nu + A^\nu\right) \tag{10}$$

$$= M^2A^\nu \tag{11}$$

The RHS of 4 can be calculated similarly. Again, you can insert the metric tensor to convert the indexes on $F^{\mu\nu}$ to lower indexes if you like, before taking the derivative. We get

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = -\frac{1}{4}\left[\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right)\right]\times 2 \tag{12}$$

$$= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \tag{13}$$

$$= -F^{\mu\nu} \tag{14}$$

where the last line follows from the antisymmetry of $F^{\mu\nu}$. Plugging this and 6 into 4 we get

$$M^2A^\nu - j^\nu = -\partial_\mu F^{\mu\nu} \tag{15}$$

$$\partial_\mu F^{\mu\nu} + M^2A^\nu = j^\nu \tag{16}$$

To obtain the propagator, we follow a similar procedure to that for the scalar field. We can write 16 as

$$\left(g^{\lambda\mu}\Box - \partial^\lambda\partial^\mu + g^{\lambda\mu}M^2\right)A_\mu = j^\lambda \tag{17}$$

The equation for the Green's function (propagator) in spacetime coordinates is then

$$\left(g^{\lambda\mu}\Box - \partial^\lambda\partial^\mu + g^{\lambda\mu}M^2\right)D_{\mu\nu}\left(x-x'\right) = g^\lambda_\nu\delta^4\left(x-x'\right) \tag{18}$$

We then define the Fourier transform as

$$D_{\mu\nu}\left(x-x'\right)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}D_{\mu\nu}\left(k\right) \tag{19}$$

Plugging this into 18 gives

$$\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}\left(-g^{\lambda\mu}k^2+k^\lambda k^\mu+g^{\lambda\mu}M^2\right)D_{\mu\nu}\left(k\right)=g_\nu^\lambda\delta^4\left(x-x'\right) \tag{20}$$

Since

$$\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}=\delta^4\left(x-x'\right) \tag{21}$$

we must have

$$\left(-g^{\lambda\mu}k^2+k^\lambda k^\mu+g^{\lambda\mu}M^2\right)D_{\mu\nu}\left(k\right)=g_\nu^\lambda \tag{22}$$

As in L&P's equation 8.18, the most general form of $D_{\mu\nu}\left(k\right)$ is

$$D_{\mu\nu}\left(k\right)=ag_{\mu\nu}+bk_\mu k_\nu \tag{23}$$

where a and b are constants. Plugging this into 22 we have

$$\left(-g^{\lambda\mu}k^2+k^\lambda k^\mu+g^{\lambda\mu}M^2\right)\left(ag_{\mu\nu}+bk_\mu k_\nu\right)=g_\nu^\lambda \tag{24}$$

$$\begin{aligned}-ag_\nu^\lambda\left(k^2-M^2\right)+ag_{\mu\nu}k^\lambda k^\mu\\ -bg^{\lambda\mu}k^2k_\mu k_\nu+bk^\lambda k^\mu k_\mu k_\nu+bg^{\lambda\mu}M^2k_\mu k_\nu=g_\nu^\lambda\end{aligned} \tag{25}$$

The term $ag_{\mu\nu}k^\lambda k^\mu$ can be rewritten as

$$ag_{\mu\nu}k^\lambda k^\mu=ak^\lambda k_\nu=ag^{\lambda\mu}k_\mu k_\nu \tag{26}$$

Remember that μ is a dummy index as it is summed, so we can redefine it to be any symbol.

Similarly

$$bk^\lambda k^\mu k_\mu k_\nu=bg^{\lambda\mu}k^2k_\mu k_\nu \tag{27}$$

so we have

$$-ag_\nu^\lambda\left(k^2-M^2\right)+g^{\lambda\mu}k_\mu k_\nu\left[a-bk^2+bk^2+bM^2\right]=g_\nu^\lambda \tag{28}$$

Equating coefficients we have

$$a = -\frac{1}{k^2 - M^2} \tag{29}$$

$$a - bk^2 + bk^2 + bM^2 = a + bM^2 = 0 \tag{30}$$

$$b = -\frac{a}{M^2} = \frac{1}{M^2\left(k^2 - M^2\right)} \tag{31}$$

Plugging this into 23 we have

$$D_{\mu\nu}\left(k\right) = \frac{1}{k^2 - M^2}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right) \tag{32}$$

Thus the propagator can be found for a field with a massive particle. However, if we let $M \to 0$ we see that $D_{\mu\nu}\left(k\right)$ blows up, so we can't get a propagator for an electromagnetic field with a zero-mass particle.

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