GAUGE-FIXING IN ELECTROMAGNETISM

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Post date: 7 Sep 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.2.

We've seen that we cannot define a propagator for the electromagnetic field from the usual equations of motion

$$\partial_\mu F^{\mu\lambda} = j^\lambda \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

is the electromagnetic field strength tensor. In an attempt to recitfy this situation, a first step is to choose a particular gauge for the electromagnetic potential A_μ. We've already seen examples of gauges in classical electromagnetism. In general, if we replace A_μ by

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta \tag{3}$$

where $\theta = \theta(x)$ is any differentiable function of space-time, then we have

$$\begin{aligned}
F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu && (4)\\
&= \partial_\mu A_\nu + \partial_\mu \partial_\nu \theta - \partial_\nu A_\mu - \partial_\nu \partial_\mu \theta && (5)\\
&= \partial_\mu A_\nu - \partial_\nu A_\mu && (6)\\
&= F_{\mu\nu} && (7)
\end{aligned}$$

since in the second line, the order in which the partial derivatives of θ are done doesn't matter.

The Lorenz gauge is given by the condition

$$\partial_\mu A^\mu = 0 \tag{8}$$

We are then led to the question of finding $\theta(x)$ such that the Lorenz gauge condition is satisfied.

Let's start with a general electromagnetic potential, so that

$$\partial_\mu A^\mu = f(x) \tag{9}$$

where $f(x)$ is some differentiable function of space-time (not necessarily zero). We now transform A_μ according to 3 and try to find the condition on θ that gives a new potential A'_μ that satisfies the Lorenz gauge condition. We have

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \theta \tag{10}$$
$$= f(x) + \Box\theta = 0 \tag{11}$$

The last line can be written as

$$\Box\theta = -f(x) \tag{12}$$

which is the wave equation with a source term $-f(x)$ and a velocity of 1 (that is, the speed of light). This equation has the same form as the Klein-Gordon equation with zero mass so we can solve it using the Green's function method we used earlier to find the Feynman propagator for a scalar field.

We define the Green's function by the condition

$$\Box_x G_0\left(x-x'\right) = -\delta^4\left(x-x'\right) \tag{13}$$

where the subscript x on $\Box_x$ indicates that the derivatives are with respect to the unprimed coordinates x.

We can see by direct substitution that θ can be expressed in terms of G_0 by

$$\theta(x) = \theta_0(x) + \int d^4x'\, G_0\left(x-x'\right) f\left(x'\right) \tag{14}$$

where

$$\Box\theta_0(x) = 0 \tag{15}$$

is a solution of the homogeneous wave equation (that is, the equation with $f(x) = 0$). This follows because

$$\Box\theta(x) = \Box\theta_0(x) + \Box\int d^4x'\, G_0(x-x')\, f(x') \tag{16}$$

$$= 0 + \int d^4x'\, \Box_x G_0(x-x')\, f(x') \tag{17}$$

$$= -\int d^4x'\, \delta^4(x-x')\, f(x') \tag{18}$$

$$= -f(x) \tag{19}$$

where in the second line, we can take the operator $\Box_x$ inside the integral since it acts only on the x coordinates which are not integrated over.