

PHOTON FIELD: FOURIER DECOMPOSITION AND POLARIZATION VECTORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.3.

I'm not 100% clear on the theory behind this topic, so the following should be read with that in mind.

The goal is to find a Fourier decomposition of the electromagnetic four-vector potential A^μ . Since A^μ is a four-vector, we can write it as a linear combination of some set of four basis vectors. The Fourier decomposition is in momentum space, so these basis vectors will depend on the four-momentum. L&P state that the appropriate decomposition is

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} \sum_{r=0}^3 \left[\epsilon_r^\mu(k) a_r(k) e^{-ik \cdot x} + \epsilon_r^{*\mu}(k) a_r^\dagger(k) e^{ik \cdot x} \right] \quad (1)$$

In this expression, $a_r(k)$ and $a_r^\dagger(k)$ are annihilation and creation operators for photons. The photon energy is the usual

$$\omega_{\mathbf{k}} = k_0 = |\mathbf{p}| \quad (2)$$

and since a photon is massless

$$k^2 = \omega_{\mathbf{k}}^2 - |\mathbf{p}|^2 = 0 \quad (3)$$

The vector content in 1 is in the objects $\epsilon_r^\mu(k)$ which are called *polarization vectors*. The index r labels which of the four vectors we're referring to, and the index μ labels the component of that vector. The sum over $r = 0 \dots 3$ is the linear combination of these vectors that produces the vector four-potential A^μ .

As we might expect, there is a lot of freedom in how we can choose the set of basis vectors ϵ_r . L&P give one convention in their section 8.5, the utility of which will presumably become more apparent as we progress, so I can't comment on their choice here. For reference, the polarization vectors are chosen to satisfy

$$\epsilon_r^\mu \epsilon_{s\mu}^* = -\zeta_r \delta_{rs} \quad (4)$$

where

$$\zeta_0 = -1 \quad (5)$$

$$\zeta_i = +1 \text{ for } i = 1, 2, 3 \quad (6)$$

The vectors satisfy a completeness relation

$$\sum_{r=0}^3 \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} = -g^{\mu\nu} \quad (7)$$

These relations apply to any set of polarization vectors.

We can now make some specific choices for the vectors. We take the four-vector n^μ to be a time-like vector satisfying

$$n^\mu n_\mu = 1 \quad (8)$$

with $n^0 > 0$. We take

$$\epsilon_0^\mu = n^\mu \quad (9)$$

Here ϵ_0 is called the *scalar polarization vector*. We then take ϵ_3^μ to lie in the $n - k$ plane (this is a 'hyperplane' in 4 dimensions) and require it to satisfy

$$\epsilon_{3\mu} n^\mu = 0 \quad (10)$$

From 4 we must also have

$$\epsilon_3 \cdot \epsilon_3 = -1 \quad (11)$$

To get an explicit formula for ϵ_3^μ we start with the fact that it lies in the $n - k$ plane, so it must be a linear combination of n^μ and k^μ . So we must have

$$\epsilon_{3\mu} = B n_\mu + C k_\mu \quad (12)$$

Applying 10 and 8 we have

$$\epsilon_{3\mu} n^\mu = B n_\mu n^\mu + C k_\mu n^\mu \quad (13)$$

$$= B + C k_\mu n^\mu = 0 \quad (14)$$

so we have

$$B = -C k_\mu n^\mu = -C k \cdot n \quad (15)$$

From 11 we have

$$e_3 \cdot e_3 = B^2 n_\mu n^\mu + C^2 k_\mu k^\mu + 2BC n \cdot k \quad (16)$$

$$= C^2 \left[(n \cdot k)^2 + k^2 - 2(n \cdot k)^2 \right] \quad (17)$$

$$= C^2 \left[k^2 - (n \cdot k)^2 \right] = -1 \quad (18)$$

giving

$$C = \frac{1}{\sqrt{(n \cdot k)^2 - k^2}} \quad (19)$$

$$B = -\frac{n \cdot k}{\sqrt{(n \cdot k)^2 - k^2}} \quad (20)$$

Putting it all together, we have

$$\epsilon_3^\mu = \frac{k^\mu - (n \cdot k) n^\mu}{\sqrt{(n \cdot k)^2 - k^2}} \quad (21)$$

ϵ_3^μ is called the *longitudinal polarization vector*. The other two vectors, ϵ_1 and ϵ_2 , are the *transverse polarization vectors*.

If we choose $n^\mu = (1, 0, 0, 0)$ and axis 3 to be along the direction of propagation of the wave, that is, along \mathbf{k} , then

$$k = (k^0, 0, 0, k^0) \quad (22)$$

For a massless photon, $k^2 = 0$ and thus $n \cdot k = k^0$, so from 21 we have

$$\epsilon_3^\mu = (0, 0, 0, 1) \quad (23)$$

The transverse vectors must then be defined to satisfy 4, which means they must be orthogonal to the $n - k$ plane, but this still gives us some freedom. One simple choice is

$$e_1^\mu = (0, 1, 0, 0) \quad (24)$$

$$e_2^\mu = (0, 0, 1, 0) \quad (25)$$

giving the complete set of basis vectors as

$$e_0^\mu = (1, 0, 0, 0) \quad (26)$$

$$e_1^\mu = (0, 1, 0, 0) \quad (27)$$

$$e_2^\mu = (0, 0, 1, 0) \quad (28)$$

$$e_3^\mu = (0, 0, 0, 1) \quad (29)$$

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