PHOTON FIELD: FOURIER DECOMPOSITION AND POLARIZATION VECTORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.3.

I'm not 100% clear on the theory behind this topic, so the following should be read with that in mind.

The goal is to find a Fourier decomposition of the electromagnetic four-vector potential A^{μ}. Since A^{μ} is a four-vector, we can write it as a linear combination of some set of four basis vectors. The Fourier decomposition is in momentum space, so these basis vectors will depend on the four-momentum. L&P state that the appropriate decomposition is

$$A^{\mu}(x)=\int\frac{d^{3}k}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}}}}\sum_{r=0}^{3}\left[\epsilon_{r}^{\mu}(k)\,a_{r}(k)\,e^{-ik\cdot x}+\epsilon_{r}^{*\mu}(k)\,a_{r}^{\dagger}(k)\,e^{ik\cdot x}\right]\tag{1}$$

In this expression, $a_{r}(k)$ and $a_{r}^{\dagger}(k)$ are annihilation and creation operators for photons. The photon energy is the usual

$$\omega_{\mathbf{k}}=k_{0}=|\mathbf{p}|\tag{2}$$

and since a photon is massless

$$k^{2}=\omega_{\mathbf{k}}^{2}-|\mathbf{p}|^{2}=0\tag{3}$$

The vector content in 1 is in the objects $\epsilon_{r}^{\mu}(k)$ which are called *polarization vectors*. The index r labels which of the four vectors we're referring to, and the index μ labels the component of that vector. The sum over $r=0\ldots3$ is the linear combination of these vectors that produces the vector four-potential A^{μ}.

As we might expect, there is a lot of freedom in how we can choose the set of basis vectors ϵ_{r}. L&P give one convention in their section 8.5, the utility of which will presumably become more apparent as we progress, so I can't comment on their choice here. For reference, the polarization vectors are chosen to satisfy

$$\epsilon_r^\mu \epsilon_{s\mu}^* = -\zeta_r \delta_{rs} \tag{4}$$

where

$$\zeta_0 = -1 \tag{5}$$

$$\zeta_i = +1 \text{ for } i = 1,2,3 \tag{6}$$

The vectors satisfy a completeness relation

$$\sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} = -g^{\mu\nu} \tag{7}$$

These relations apply to any set of polarization vectors.

We can now make some specific choices for the vectors. We take the four-vector n^μ to be a time-like vector satisfying

$$n^\mu n_\mu = 1 \tag{8}$$

with $n^0 > 0$. We take

$$\epsilon_0^\mu = n^\mu \tag{9}$$

Here ϵ_0 is called the *scalar polarization vector*. We then take ϵ_3^μ to lie in the $n-k$ plane (this is a 'hyperplane' in 4 dimensions) and require it to satisfy

$$\epsilon_{3\mu} n^\mu = 0 \tag{10}$$

From 4 we must also have

$$\epsilon_3 \cdot \epsilon_3 = -1 \tag{11}$$

To get an explicit formula for ϵ_3^μ we start with the fact that it lies in the $n-k$ plane, so it must be a linear combination of n^μ and k^μ. So we must have

$$\epsilon_{3\mu} = Bn_\mu + Ck_\mu \tag{12}$$

Applying 10 and 8 we have

$$\epsilon_{3\mu} n^\mu = Bn_\mu n^\mu + Ck_\mu n^\mu \tag{13}$$

$$= B + Ck_\mu n^\mu = 0 \tag{14}$$

so we have

$$B = -Ck_\mu n^\mu = -Ck \cdot n \tag{15}$$

From 11 we have

$$e_3 \cdot e_3 = B^2 n_\mu n^\mu + C^2 k_\mu k^\mu + 2BCn \cdot k \tag{16}$$
$$= C^2 \left[(n \cdot k)^2 + k^2 - 2(n \cdot k)^2 \right] \tag{17}$$
$$= C^2 \left[k^2 - (n \cdot k)^2 \right] = -1 \tag{18}$$

giving

$$C = \frac{1}{\sqrt{(n \cdot k)^2 - k^2}} \tag{19}$$
$$B = -\frac{n \cdot k}{\sqrt{(n \cdot k)^2 - k^2}} \tag{20}$$

Putting it all together, we have

$$\epsilon_3^\mu = \frac{k^\mu - (n \cdot k) n^\mu}{\sqrt{(n \cdot k)^2 - k^2}} \tag{21}$$

ϵ_3^μ is called the *longitudinal polarization vector*. The other two vectors, ϵ_1 and ϵ_2, are the *transverse polarization vectors*.

If we choose $n^\mu = (1,0,0,0)$ and axis 3 to be along the direction of propagation of the wave, that is, along **k**, then

$$k = \left(k^0, 0, 0, k^0\right) \tag{22}$$

For a massless photon, $k^2 = 0$ and thus $n \cdot k = k^0$, so from 21 we have

$$\epsilon_3^\mu = (0,0,0,1) \tag{23}$$

The transverse vectors must then be defined to satisfy 4, which means they must be orthogonal to the $n-k$ plane, but this still gives us some freedom. One simple choice is

$$e_1^\mu = (0,1,0,0) \tag{24}$$
$$e_2^\mu = (0,0,1,0) \tag{25}$$

giving the complete set of basis vectors as

$$e_0^\mu = (1,0,0,0) \tag{26}$$
$$e_1^\mu = (0,1,0,0) \tag{27}$$
$$e_2^\mu = (0,0,1,0) \tag{28}$$
$$e_3^\mu = (0,0,0,1) \tag{29}$$

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