PHOTON FIELD COMMUTATORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.4.

The classical Lagrangian for a pure electric field (that is, without any currents) is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

where $F_{\mu\nu}$ can be written in terms of the 4-potential $A_\mu$as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

As discussed by L&P in Chapter 8, it is not possible to create a propagator using this Lagrangian as it stands, so some modification is needed. The problem is that there is too much freedom in the choice of the four-potential A_μ. To fix this problem, the classical Lagrangian is modified by adding a gauge-fixing term

$$\mathscr{L}_{\text{GF}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \tag{3}$$

where ξ is a parameter that can be chosen to fix the particular gauge. The full Lagrangian is now

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \tag{4}$$

The momentum that is conjugate to the field component A_μ is

$$\Pi^\mu = \frac{\delta\mathscr{L}}{\delta\left(\partial_0 A_\mu\right)} \tag{5}$$

$$= -\dot{A}^\mu + \left(1 - \frac{1}{\xi}g^{\mu 0}\partial_\nu A^\nu\right) \tag{6}$$

L&P use the gauge defined by

$$\xi = 1 \tag{7}$$

so the canonical momenta become

$$\Pi^{\mu} = -\dot{A}^{\mu} \tag{8}$$

The usual commutation relations between a field and a momentum is given by L&P's eqn 8.32:

$$\left[A_{\mu}(t,\mathbf{x}), \Pi^{\nu}(t,\mathbf{y})\right] = i\delta_{\mu}^{\nu}\delta^{3}(\mathbf{x}-\mathbf{y}) \tag{9}$$

We can relate this commutator to the commutators for the creation and annihilation operators introduced in the Fourier decomposition:

$$A^{\mu}(t,\mathbf{x}) = \int \frac{d^{3}k}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}}}} \sum_{r=0}^{3}\left[\epsilon_{r}^{\mu}(k)\,a_{r}(k)\,e^{-ik\cdot x} + \epsilon_{r}^{*\mu}(k)\,a_{r}^{\dagger}(k)\,e^{ik\cdot x}\right] \tag{10}$$

Taking the time derivative, we have (where we've introduced the integration variable k' and summation variable s to distinguish Π^{ν} from A^{μ}.

$$\begin{aligned}\Pi^{\nu}(t,\mathbf{y}) &= -\partial_{0}A^{\nu} && (11)\\ &= i\int \frac{d^{3}k'}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}'}}}\omega_{\mathbf{k}'}\sum_{s=0}^{3}\left[\epsilon_{s}^{\nu}(k')\,a_{s}(k')\,e^{-ik'\cdot y} - \epsilon_{s}^{*\nu}(k')\,a_{s}^{\dagger}(k')\,e^{ik'\cdot y}\right] && (12)\end{aligned}$$

We can now work out the commutator 9 by direct (although messy) calculation. Converting the index μ to an upper index, we get

$$\begin{aligned}\left[A^{\mu}(t,\mathbf{x}), \Pi^{\nu}(t,\mathbf{y})\right] = \frac{i}{2(2\pi)^{3}}\int\frac{d^{3}k}{\sqrt{\omega_{\mathbf{k}}}}\int\frac{d^{3}k\,\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}'}}}\sum_{r,s}\Big\{&\epsilon_{r}^{\mu}(k)\,\epsilon_{s}^{\nu}(k')\,e^{-i(k\cdot x+k'\cdot y)}\left[a_{r}(k), a_{s}(k')\right]\\ &-\epsilon_{r}^{\mu}(k)\,\epsilon_{s}^{*\nu}(k')\,e^{-i(k\cdot x-k'\cdot y)}\left[a_{r}(k), a_{s}^{\dagger}(k')\right]\\ &+\epsilon_{r}^{*\mu}(k)\,\epsilon_{s}^{\nu}(k')\,e^{i(k\cdot x-k'\cdot y)}\left[a_{r}^{\dagger}(k), a_{s}(k')\right]\\ &-\epsilon_{r}^{*\mu}(k)\,\epsilon_{s}^{*\nu}(k')\,e^{i(k\cdot x+k'\cdot y)}\left[a_{r}^{\dagger}(k), a_{s}^{\dagger}(k')\right]\Big\}\end{aligned} \tag{13}$$

We require this to be equal to the RHS of 9, so to get the delta function $\delta^{3}(\mathbf{x}-\mathbf{y})$ we need to get rid of terms involving $e^{\pm i(k\cdot x+k'\cdot y)}$. We can do this by requiring

$$\left[a_r(k), a_s(k')\right] = 0 \tag{14}$$

$$\left[a_r^\dagger(k), a_s^\dagger(k')\right] = 0 \tag{15}$$

Inserting these into 13 gets rid of the first and last lines. Flipping the commutator in the third line gives us

$$[A^\mu(t,\mathbf{x}), \Pi^\nu(t,\mathbf{y})] = \frac{-i}{2(2\pi)^3}\int \frac{d^3k}{\sqrt{\omega_\mathbf{k}}}\int \frac{d^3k\ \omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}'}}}\sum_{r,s}\left\{\epsilon_r^\mu(k)\,\epsilon_s^{*\nu}(k')\,e^{-i(k\cdot x-k'\cdot y)}\left[a_r(k), a_s^\dagger(k')\right]\right.$$
$$\left.+\epsilon_r^{*\mu}(k)\,\epsilon_s^\nu(k')\,e^{i(k\cdot x-k'\cdot y)}\left[a_s(k'), a_r^\dagger(k)\right]\right\} \tag{16}$$

If we now impose the condition given by L&P's eqn 8.56:

$$\left[a_r(k), a_s^\dagger(k')\right] = \zeta_r\delta_{rs}\delta^3(\mathbf{k}-\mathbf{k}') \tag{17}$$

where

$$\zeta_0 = -1 \tag{18}$$

$$\zeta_i = +1 \text{ for } i = 1,2,3 \tag{19}$$

we have

$$[A^\mu(t,\mathbf{x}), \Pi^\nu(t,\mathbf{y})] = \frac{-i}{2(2\pi)^3}\int \frac{d^3k}{\sqrt{\omega_\mathbf{k}}}\int \frac{d^3k\ \omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}'}}}\sum_{r,s}\zeta_r\delta_{rs}\delta^3(\mathbf{k}-\mathbf{k}')$$
$$\times\left\{\epsilon_r^\mu(k)\,\epsilon_s^{*\nu}(k')\,e^{-i(k\cdot x-k'\cdot y)}\right. \tag{20}$$
$$\left.+\epsilon_r^{*\mu}(k)\,\epsilon_s^\nu(k')\,e^{i(k\cdot x-k'\cdot y)}\right\} \tag{21}$$

We can now do one of the integrals, say over $\mathbf{k}'$, by using the delta function, to get

$$[A^\mu(t,\mathbf{x}), \Pi^\nu(t,\mathbf{y})] = \frac{-i}{2(2\pi)^3}\int d^3k\sum_{r,s}\zeta_r\delta_{rs}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\epsilon_r^\mu\epsilon_s^{*\nu}+\epsilon_r^{*\mu}\epsilon_s^\nu\right\} \tag{22}$$

We've omitted the explicit dependence of the ϵs on k since they now all depend on the same k. Also, since both spacetime points x and y refer to the same time, $x_0 = y_0$ so $e^{-ik\cdot(x-y)} = e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$. Since we're integrating over all k, the exponential factors $e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ and $e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ are equivalent under the integral.

We now take advantage of the δ_{rs} factor to get

$$\left[A^{\mu}\left(t,\mathbf{x}\right),\Pi^{\nu}\left(t,\mathbf{y}\right)\right]=\frac{-i}{2\left(2\pi\right)^{3}}\int d^{3}k\sum_{r}\zeta_{r}e^{-i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)}\left\{\epsilon_{r}^{\mu}\epsilon_{r}^{*\nu}+\epsilon_{r}^{*\mu}\epsilon_{r}^{\nu}\right\} \tag{23}$$

We can use the completeness formula

$$\sum_{r=0}^{3}\zeta_{r}\epsilon_{r}^{\mu}\epsilon_{r}^{*\nu}=-g^{\mu\nu} \tag{24}$$

to get

$$\left[A^{\mu}\left(t,\mathbf{x}\right),\Pi^{\nu}\left(t,\mathbf{y}\right)\right]=\frac{-i}{2\left(2\pi\right)^{3}}\int d^{3}k\;e^{-i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)}\left(-g^{\mu\nu}-g^{\nu\mu}\right) \tag{25}$$

$$=ig^{\mu\nu}\frac{1}{\left(2\pi\right)^{3}}\int d^{3}k\;e^{-i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)} \tag{26}$$

$$=ig^{\mu\nu}\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{27}$$

If we lower the index μ we get 9 again:

$$\left[A_{\mu}\left(t,\mathbf{x}\right),\Pi^{\nu}\left(t,\mathbf{y}\right)\right]=i\delta_{\mu}^{\nu}\delta^{3}\left(\mathbf{x}-\mathbf{y}\right) \tag{28}$$

Thus the commutators 14 and 17 together are consistent with the original commutator 9.

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