PHOTON FIELD MOMENTUM

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercises 8.5 - 8.6.

Greiner & Reinhardt, *Field Quantization*, Chapter 7.

In their section 8.3, L&P show that the Lagrangian for the photon field can be rewritten as

$$\mathscr{L}=-\frac{1}{2}\left(\partial_{\mu}A_{\nu}\right)\left(\partial^{\mu}A^{\nu}\right)+\frac{1}{2}\left(1-\frac{1}{\xi}\right)\left(\partial_{\mu}A^{\mu}\right)^{2} \tag{1}$$

where the last term incorporates the gauge fixing term

$$\mathscr{L}_{\text{GF}}=-\frac{1}{2\xi}\left(\partial_{\mu}A^{\mu}\right)^{2} \tag{2}$$

The field operators A^{μ} can be written as Fourier decompositions

$$A^{\mu}\left(x\right)=\int\frac{d^{3}k}{\sqrt{\left(2\pi\right)^{3}2\omega_{\mathbf{k}}}}\sum_{r=0}^{3}\left[\epsilon_{r}^{\mu}\left(k\right)a_{r}\left(k\right)e^{-ik\cdot x}+\epsilon_{r}^{*\mu}\left(k\right)a_{r}^{\dagger}\left(k\right)e^{ik\cdot x}\right] \tag{3}$$

We now wish to find the components of the four-momentum P^{μ} in terms of creation and annihilation operators. The four-momentum is defined in terms of the stress-energy tensor by L&P's eqn 2.55:

$$P^{\mu}=\int d^{3}x\,T^{0\mu} \tag{4}$$

where

$$T^{0\mu}=\frac{\partial\mathscr{L}}{\partial\dot{A}^{\rho}}\partial^{\mu}A^{\rho}-g^{0\mu}\mathscr{L} \tag{5}$$

From 1 we have

$$\frac{\partial\mathscr{L}}{\partial\dot{A}^{\rho}}=-\dot{A}_{\rho}+\left(1-\frac{1}{\xi}\right)g_{\rho0}\partial_{\nu}A^{\nu} \tag{6}$$

Let's consider the spatial components P^j first (where $j = 1,2,3$). From 5 and 6, we have

$$T^{0j} = \frac{\partial \mathscr{L}}{\partial \dot{A}^\rho} \partial^j A^\rho \tag{7}$$

$$= -\dot{A}_\rho \partial^j A^\rho + \left(1 - \frac{1}{\xi}\right) (\partial_\nu A^\nu) \partial^j A^0 \tag{8}$$

We are interested in terms that give physically meaningful results, and for this we need to look at the Gupta-Bleuler condition (L&P section 8.6) which states that for any two physical states $|\Psi\rangle$ and $|\Psi'\rangle$, we must have

$$\left\langle \Psi' \left| \partial_\nu A^\nu \right| \Psi \right\rangle = 0 \tag{9}$$

Taking the matrix element of the second term in 8, we can insert a complete set of states to get

$$\left(1 - \frac{1}{\xi}\right) \left\langle \Psi' \left| (\partial_\nu A^\nu) \partial^j A^0 \right| \Psi \right\rangle = \left(1 - \frac{1}{\xi}\right) \int dk \, \left\langle \Psi' \left| \partial_\nu A^\nu \right| k \right\rangle \left\langle k \left| \partial^j A^0 \right| \Psi \right\rangle \tag{10}$$

$$= 0 \tag{11}$$

where the conclusion follows because we must have $\langle \Psi' | \partial_\nu A^\nu | k \rangle = 0$ for all states. We can therefore restrict our attention to the first term in 8 and use that to work out P^μ. Therefore we have

$$P^\mu = -\int d^3x \, \dot{A}_\rho \partial^j A^\rho \tag{12}$$

We can now use 3 to convert to creation and annihilation operators. We have

$$\dot{A}_\rho = -i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} \omega_{\mathbf{k}} \sum_{r=0}^{3} \left[\epsilon_{\rho r}(k) a_r(k) e^{-ik\cdot x} - \epsilon^*_{\rho r}(k) a^\dagger_r(k) e^{ik\cdot x} \right] \tag{13}$$

$$\partial^j A^\rho = i \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\mathbf{p}}}} p^j \sum_{s=0}^{3} \left[\epsilon^\rho_s(p) a_s(p) e^{-ip\cdot x} - \epsilon^{*\rho}_s(p) a^\dagger_s(p) e^{ip\cdot x} \right] \tag{14}$$

I've used k for momentum in the first equation and p in the second (and r and s for the summation indexes) to avoid mixing them up when we combine them. To get the sign in these equations, remember that

$$k\cdot x = g_{\mu\nu}k^{\mu}x^{\nu} \tag{15}$$
$$= k_0x^0 - k_jx^j \tag{16}$$
$$= \omega_{\mathbf{k}}t - k_jx^j \tag{17}$$

When we multiply these two integrals together and integrate over d^3x as in 4, we will get delta functions involving k and p. Let's consider the first term in 13 (which we'll call ①) multiplied by the first term in 14 (③):

$$① \times ③ = \int \frac{d^3k}{\sqrt{2\omega_{\mathbf{k}}}}\omega_{\mathbf{k}} \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}p^j \times \sum_{r,s}\epsilon_{\rho r}(k)\,a_r(k)\,\epsilon_s^{\rho}(p)\,a_s(p) \times e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})t}\frac{1}{(2\pi)^3}\int d^3x\; e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} \tag{18}$$

The integral in the last line is

$$\frac{1}{(2\pi)^3}\int d^3x\; e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} = \delta^3(\mathbf{k}+\mathbf{p}) \tag{19}$$

so $\mathbf{k} = -\mathbf{p}$, $\omega_{\mathbf{k}} = \omega_{\mathbf{p}}$ and we have the result

$$① \times ③ = \frac{1}{2}\int d^3k\; k^j \sum_{r,s}\epsilon_{\rho r}(k)\,a_r(k)\,\epsilon_s^{\rho}(-k)\,a_s(-k)\,e^{-2i\omega_{\mathbf{k}}t} \tag{20}$$

Due to the k_j in the integrand and the fact that $\sum_{r,s}\epsilon_{\rho r}(k)\,a_r(k)\,\epsilon_s^{\rho}(-k)\,a_s(-k)\,e^{-2i\omega_{\mathbf{k}}t}$ is an even function of k, the integral over k_j gives zero. The same is true for the product of the second term in 13 (②) and the second term in 14 (④). We are therefore left with the other two products. For ② × ③ we have

$$② \times ③ = -\int \frac{d^3k}{\sqrt{2\omega_{\mathbf{k}}}}\omega_{\mathbf{k}} \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}}p^j \times \sum_{r,s}\epsilon^*_{\rho r}(k)\,a_r^{\dagger}(k)\,\epsilon_s^{\rho}(p)\,a_s(p) \times e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{p}})t}\frac{1}{(2\pi)^3}\int d^3x\; e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} \tag{21}$$

Now the delta function sets $\mathbf{k} = \mathbf{p}$ so we have

$$\textcircled{2}\times\textcircled{3} = -\frac{1}{2}\int d^3k\; k^j \sum_{r,s}\epsilon^*_{\rho r}\epsilon^\rho_s a^\dagger_r a_s \tag{22}$$

where we've omitted the explicit dependence on k since all terms have the same dependence. We can now use the relation

$$\epsilon^\mu_r \epsilon^*_{s\mu} = -\zeta_r \delta_{rs} \tag{23}$$

to get

$$\textcircled{2}\times\textcircled{3} = \frac{1}{2}\int d^3k\; k^j \sum_r \zeta_r a^\dagger_r a_r \tag{24}$$

The other product gives a similar result, except the order of the $a^\dagger$ and a operators is reversed:

$$\textcircled{1}\times\textcircled{4} = \frac{1}{2}\int d^3k\; k^j \sum_r \zeta_r a_r a^\dagger_r \tag{25}$$

If we impose normal ordering, we get

$$P^j =: \frac{1}{2}\int d^3k\; k^j \sum_r \zeta_r a^\dagger_r a_r + \frac{1}{2}\int d^3k\; k^j \sum_r \zeta_r a_r a^\dagger_r : \tag{26}$$

$$= \int d^3k\; k^j \sum_r \zeta_r a^\dagger_r a_r \tag{27}$$

$$= \int d^3k\; k^j \left(\sum_{r=1}^{3} a^\dagger_r a_r - a^\dagger_0 a_0\right) \tag{28}$$

Imposing the Gupta-Bleuler condition requires that

$$a_0 \left|\Psi\right\rangle = a_3 \left|\Psi\right\rangle \tag{29}$$

so two terms in the sum cancel and we're left with

$$P^j = \int d^3k\; k^j \sum_{r=1}^{2} a^\dagger_r(k)\, a_r(k) \tag{30}$$

Now for P^0. From 5 we have

$$T^{00} = \left[-\dot{A}_\rho + \left(1-\frac{1}{\xi}\right) g_{\rho 0}\partial_\nu A^\nu\right]\dot{A}^\rho + \frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) - \frac{1}{2}\left(1-\frac{1}{\xi}\right)\left(\partial_\mu A^\mu\right)^2 \tag{31}$$

Invoking the condition 9 again, we can dispose of the terms involving $\partial_\nu A^\nu$ so we're left with

$$T^{00} = -\dot{A}_\rho \dot{A}^\rho + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \tag{32}$$

$$= -\dot{A}_\rho \dot{A}^\rho + \frac{1}{2}\dot{A}_\rho \dot{A}^\rho + \frac{1}{2}(\partial_j A_\nu)\left(\partial^j A^\nu\right) \tag{33}$$

$$= -\frac{1}{2}\dot{A}_\rho \dot{A}^\rho - \frac{1}{2}\nabla A \cdot \nabla A \tag{34}$$

When we now substitute using 13 and 14 we find that terms ① × ③ and ② × ④ contain a factor of $\omega_{\mathbf{k}}^2 - \mathbf{k}\cdot\mathbf{k}$ after applying the delta function. For the massless photon, we have

$$\omega_{\mathbf{k}}^2 - \mathbf{k}\cdot\mathbf{k} = 0 \tag{35}$$

so again, these terms vanish. For ① × ④ and ② × ③, the delta function gives us a factor of $\omega_{\mathbf{k}}^2 + \mathbf{k}\cdot\mathbf{k} = 2\omega_{\mathbf{k}}^2$ so we get, after following essentially the same steps as for P^j above,

$$P^0 =: \frac{1}{2}\int d^3k\ \omega_{\mathbf{k}} \sum_r \zeta_r a_r^\dagger a_r + \frac{1}{2}\int d^3k\ \omega_{\mathbf{k}} \sum_r \zeta_r a_r a_r^\dagger : \tag{36}$$

$$= \int d^3k\ \omega_{\mathbf{k}} \sum_r \zeta_r a_r^\dagger a_r \tag{37}$$

$$= \int d^3k\ \omega_{\mathbf{k}} \left(\sum_{r=1}^{3} a_r^\dagger a_r - a_0^\dagger a_0\right) \tag{38}$$

$$= \int d^3k\ \omega_{\mathbf{k}} \sum_{r=1}^{2} a_r^\dagger(k)\, a_r(k) \tag{39}$$

Thus in all cases, the two physical modes for a photon moving in the z direction are the two transverse modes with indexes $r = 1, 2$.