PHOTON FIELD: COULOMB PROPAGATOR

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 8, Exercise 8.7.

In their section 8.7, L&P show that the photon propagator can be written as

$$D^{\mu\nu}(k)=\frac{1}{k^2+i\epsilon}\left[\sum_{r-1}^{2}\epsilon_r^{\mu}(k)\,\epsilon_r^{\nu}(k)+D_C^{\mu\nu}(k)+D_R^{\mu\nu}(k)\right]\tag{1}$$

where D_C is the Coulomb part of the propagator and D_R is the remainder part.

In the frame where

$$\epsilon_0^{\mu}=n^{\mu}=(1,0,0,0)\tag{2}$$

$$\epsilon_3^{\mu}=\frac{k^{\mu}-(k\cdot n)\,n^{\mu}}{\sqrt{(k\cdot n)^2-k^2}}\tag{3}$$

we can write the Fourier transform of $D_C^{\mu\nu}(k)$ as

$$D_C^{\mu\nu}\left(x-x'\right)=\int\frac{d^3k}{(2\pi)^3}\frac{g^{\mu0}g^{\nu0}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\delta\left(x^0-x'^0\right)\tag{4}$$

To do the remaining integral, we need another result. For some parameter λ, we look at the following Fourier transform.

$$\frac{e^{-r/\lambda}}{4\pi r}=\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\;e^{i\mathbf{p}\cdot\mathbf{x}}f(\mathbf{p})\tag{5}$$

where $f(p)$ is the weighting function in the Fourier transform. We can find $f(\mathbf{p})$ in the usual way, by multiplying through by $e^{-i\mathbf{q}\cdot\mathbf{x}}$ and integrating:

$$\frac{1}{(2\pi)^3}\int d^3x\int d^3\mathbf{p}\;e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}f(\mathbf{p})=\frac{1}{4\pi}\int d^3x\;e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{e^{-r/\lambda}}{r}\tag{6}$$

The integral over x on the LHS gives us a delta function, so we have

$$\frac{1}{(2\pi)^3}\int d^3x\int d^3\mathbf{p}\,e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}f(\mathbf{p}) = \int d^3\mathbf{p}\,\delta^3(\mathbf{p}-\mathbf{q})\,f(\mathbf{p}) \tag{7}$$

$$= f(\mathbf{q}) \tag{8}$$

On the RHS of 6, we have $d^3x = r^2\sin\theta dr\ d\theta\ d\phi$. We can choose the z axis to be along $\mathbf{q}$ so that $\mathbf{q}\cdot\mathbf{x} = \mathfrak{q}r\cos\theta$ where $\mathfrak{q}\equiv|\mathbf{q}|$. The integral over ϕ gives 2π so we have

$$\frac{1}{4\pi}\int d^3x\ e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{e^{-r/\lambda}}{r} = \frac{1}{4\pi}\int r^2\sin\theta dr\ d\theta\ d\phi\ e^{-i\mathfrak{q}r\cos\theta}\frac{e^{-r/\lambda}}{r} \tag{9}$$

$$= \frac{1}{2}\int_0^\infty dr\int_0^\pi d\theta\ e^{-i\mathfrak{q}r\cos\theta}\sin\theta\ re^{-r/\lambda} \tag{10}$$

The integral over θ gives

$$\int_0^\pi d\theta\ e^{-i\mathfrak{q}r\cos\theta}\sin\theta = \frac{1}{i\mathfrak{q}r}\ e^{-i\mathfrak{q}r\cos\theta}\Big|_0^\pi \tag{11}$$

$$= \frac{1}{i\mathfrak{q}r}\left(e^{i\mathfrak{q}r} - e^{-i\mathfrak{q}r}\right) \tag{12}$$

$$= \frac{2\sin(\mathfrak{q}r)}{\mathfrak{q}r} \tag{13}$$

Putting this into 10 we have

$$\frac{1}{4\pi}\int d^3x\ e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{e^{-r/\lambda}}{r} = \frac{1}{\mathfrak{q}}\int_0^\infty dr\ e^{-r/\lambda}\sin(\mathfrak{q}r) \tag{14}$$

This integral can be looked up in tables (or, by hand, by integrating by parts twice), but I used Maple to get the result.

$$\int_0^\infty dr\ e^{-r/\lambda}\sin(\mathfrak{q}r) = -\left.\frac{\lambda e^{-r/\lambda}\left(\sin(\mathfrak{q}r) + \mathfrak{q}\lambda\cos(\mathfrak{q}r)\right)}{1+\lambda^2\mathfrak{q}^2}\right|_{r=0}^{\infty} \tag{15}$$

$$= \frac{\mathfrak{q}\lambda^2}{1+\lambda^2\mathfrak{q}^2} \tag{16}$$

If we now take the limit $\lambda\to\infty$ we have

$$\lim_{\lambda\to\infty}\frac{\mathfrak{q}\lambda^2}{1+\lambda^2\mathfrak{q}^2} = \frac{1}{\mathfrak{q}} \tag{17}$$

Inserting this into 14 we have

$$\int d^3x\,\frac{e^{-\mathbf{q}\cdot\mathbf{x}}}{4\pi r} = \lim_{\lambda\to\infty}\frac{1}{4\pi}\int d^3x\,e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{e^{-r/\lambda}}{r} \tag{18}$$

$$= \frac{1}{\mathsf{q}^2} \tag{19}$$

We can now insert this into 4. To avoid mixing up the coordinates, we'll rewrite 19 as

$$\int d^3y\,\frac{e^{-\mathbf{k}\cdot\mathbf{y}}}{4\pi y} = \frac{1}{\mathbf{k}^2} \tag{20}$$

We have

$$\int \frac{d^3k}{(2\pi)^3}\frac{g^{\mu0}g^{\nu0}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \int\frac{d^3k}{(2\pi)^3}g^{\mu0}g^{\nu0}\int d^3y\,\frac{e^{-\mathbf{k}\cdot\mathbf{y}}}{4\pi y}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{21}$$

$$= g^{\mu0}g^{\nu0}\int d^3y\frac{1}{4\pi y}\int\frac{d^3k}{(2\pi)^3}e^{\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'-\mathbf{y})} \tag{22}$$

The second integral is a delta function which sets $\mathbf{y}=\mathbf{x}-\mathbf{x}'$ so we end up with

$$\int \frac{d^3k}{(2\pi)^3}\frac{g^{\mu0}g^{\nu0}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = g^{\mu0}g^{\nu0}\int d^3y\frac{1}{4\pi y}\int\delta^3\left(\mathbf{x}-\mathbf{x}'-\mathbf{y}\right) \tag{23}$$

$$= \frac{g^{\mu0}g^{\nu0}}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|} \tag{24}$$

Inserting back into 4 we have

$$D_C^{\mu\nu}\left(x-x'\right) = \frac{g^{\mu0}g^{\nu0}}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left(x^0-x'^0\right) \tag{25}$$

The Coulomb propagator has the $1/\left|\mathbf{x}-\mathbf{x}'\right|$ dependence of the classical Coulomb potential for two charges separated by the distance $\left|\mathbf{x}-\mathbf{x}'\right|$.