LOCAL GAUGE INVARIANCE LEADING TO QUANTUM ELECTRODYNAMICS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercises 9.1-9.2.

To introduce quantum electrodynamics, L&P in their section 9.1 examine the phase rotation of a field operator under which the operator transforms as

$$\phi\rightarrow\phi^{\prime}=e^{-ieQ\theta}\phi \tag{1}$$

where e is the elementary charge (the charge on a proton) and Q is an integer, so that eQ is the charge on the particle represented by the field ϕ. If θ is a global parameter, that is, it doesn't depend on the spacetime position, then the Lagrangians for both the complex scalar field and the Dirac field are invariant, because the field operators always occur in the combination $\phi^{\dagger}\phi$ or $\psi^{\dagger}\psi$, or similar terms involving derivatives of the operators. The fact that the Lagrangians remain invariant leads, via Noether's theorem, to a conserved current which, for the Dirac Lagrangian is given by L&P's equation 9.5:

$$j^{\mu}=eQ\overline{\psi}\gamma^{\mu}\psi \tag{2}$$

L&P then require that the symmetry 1 holds locally, which means that the Lagrangian remains unchanged even if $\theta=\theta\left(x\right)$, so that it *does* depend on its location in spacetime. Quite why they do this is not explained, except that it leads to the theory of quantum electrodynamics (QED). They show, in their equations 9.8 through 9.11, that we can impose this local symmetry provided the Dirac Lagrangian is modified so that it now reads

$$\mathscr{L}=\overline{\psi}\left(i\partial\!\!\!/-m\right)\psi-eQ\overline{\psi}\gamma^{\mu}\psi A_{\mu} \tag{3}$$

If we now make the local transformation of both ψ and A_{μ} according to

$$\psi\rightarrow\psi^{\prime}=e^{-ieQ\theta}\psi \tag{4}$$

$$A_{\mu}\rightarrow A_{\mu}^{\prime}=A_{\mu}+\partial_{\mu}\theta \tag{5}$$

then the Lagrangian remains the same, so that

$$\begin{aligned}
\mathscr{L}' &= \overline{\psi}'\left(i\partial\!\!\!/ - m\right)\psi' - eQ\overline{\psi}'\gamma^{\mu}\psi' A'_{\mu} && (6)\\
&= \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - eQ\overline{\psi}\gamma^{\mu}\psi A_{\mu} && (7)\\
&= \mathscr{L} && (8)
\end{aligned}$$

We now note that the transformation 5 is the same as that under which the electromagnetic field tensor remains invariant, if A_μ is interpreted as the electromagnetic field operator. That is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{9}$$

so

$$\begin{aligned}
F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu && (10)\\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta && (11)
\end{aligned}$$

Since the order of partial differentiation is irrelevant, the last two terms cancel and we have

$$F'_{\mu\nu} = F_{\mu\nu} \tag{12}$$

The basic Lagrangian for QED is taken to be 3 with the term for the free electromagnetic field added in, so we have

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\overline{\psi}\gamma^{\mu}\psi A_{\mu} \tag{13}$$

Although this Lagrangian was obtained by combining the Dirac Lagrangian with electromagnetic one, the result is actually a bit more general. To see this, we rewrite the Lagrangian 13 as

$$\mathscr{L} = \overline{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{14}$$

where D_μ is called the gauge covariant derivative, and is defined as

$$D_\mu \equiv \partial_\mu + ieQA_\mu \tag{15}$$

We can look at the way D_μ transforms for some general field ψ (not necessarily a Dirac field), provided that ψ transforms like 4 and A_μ transforms like 5. We get

$$\begin{aligned}
D'_\mu\psi' &= \left(\partial_\mu + ieQA'_\mu\right)\psi' && (16)\\
&= \left(\partial_\mu + ieQ\left(A_\mu + \partial_\mu\theta\right)\right)e^{-ieQ\theta}\psi && (17)\\
&= e^{-ieQ\theta}\left[-ieQ\psi\partial_\mu\theta + \partial_\mu\psi + ieQA_\mu\psi + ieQ\psi\partial_\mu\theta\right] && (18)\\
&= e^{-ieQ\theta}\left(\partial_\mu + ieQA_\mu\right)\psi && (19)\\
&= e^{-ieQ\theta}D_\mu\psi && (20)
\end{aligned}$$

Thus if the Lagrangian contains terms of the form $(D_\mu\psi)^\dagger\psi$ or $(D^\mu\psi)^\dagger(D_\mu\psi)$ (in addition to terms like $\psi^\dagger\psi$ that don't involve derivatives), it will be invariant under the transformations 4 and 5. In particular, the Lagrangian of scalar electrodynamics given as equation 9.16 in L&P is:

$$\mathscr{L} = (D^\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{21}$$

is invariant under this transformation, since we've seen that the last two terms are already invariant, and

$$\begin{aligned}
\left(D'^\mu\phi'\right)^\dagger &= \left(e^{-ieQ\theta}D^\mu\phi\right)^\dagger && (22)\\
&= e^{ieQ\theta}\left(D^\mu\phi\right)^\dagger && (23)
\end{aligned}$$

so

$$\begin{aligned}
\left(D'^\mu\phi'\right)^\dagger\left(D'_\mu\phi'\right) &= e^{ieQ\theta}\left(D^\mu\phi\right)^\dagger e^{-ieQ\theta}D_\mu\phi && (24)\\
&= \left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right) && (25)
\end{aligned}$$

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