QUANTUM ELECTRODYNAMICS: NO FIRST-ORDER PROCESSES ARE POSSIBLE

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercise 9.3.

The Lagrangian for a system consisting of the electromagnetic field and fermion fields is given by

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\overline{\psi}\gamma^{\mu}\psi A_{\mu} \tag{1}$$

The first term is the Lagrangian for the Dirac field and the second term is that for a free electromagnetic field. The last term involves both the Dirac spinors ψ and the electromagnetic field A_μ, so represents an interaction between these two types of field. For the special case of the electron field, the charge is $-e$ so $Q = -1$ and the interaction Lagrangian is

$$\mathscr{L}_{\text{int}} = e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \tag{2}$$

The corresponding interaction Hamiltonian is obtained from the usual formula

$$\mathscr{H}_I = \sum \pi\dot{\phi} - \mathscr{L}_{\text{int}} \tag{3}$$

$$= -e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \tag{4}$$

Imposing the usual requirement of normal ordering we have

$$:\mathscr{H}_I: = -e:\overline{\psi}\gamma^{\mu}\psi A_{\mu}: \tag{5}$$

To use this hamiltonian to calculate S-matrix elements, we look at first order terms first, and then higher order terms if necessary. For anything beyond first order, we need Wick's theorem to convert a time-ordered product into a sum over normal ordered products.

For first order, however, the S-matrix element is

$$S_{fi}^{(1)} = \int d^4x \ \langle f\,|\mathscr{H}_I|\,i\rangle \tag{6}$$

where i and f represent the initial and final states.

The various fields in the interaction Hamiltonian 5 can be written as the sum of a creation and annihilation parts. For the photon field we have

$$A^{\mu}(x)=\int\frac{d^{3}k}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}}}}\sum_{r=0}^{3}\left[\epsilon_{r}^{\mu}(k)a_{r}(k)e^{-ik\cdot x}+\epsilon_{r}^{*\mu}(k)a_{r}^{\dagger}(k)e^{ik\cdot x}\right] \tag{7}$$

which we can write as

$$A^{\mu}(x)=A_{+}^{\mu}(x)+A_{-}^{\mu}(x) \tag{8}$$

where A_{+}^{μ} contains the annihilation operators and A_{-}^{μ} the creation operators:

$$A_{+}^{\mu}(x)=\int\frac{d^{3}k}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}}}}\sum_{r=0}^{3}\epsilon_{r}^{\mu}(k)a_{r}(k)e^{-ik\cdot x} \tag{9}$$

$$A_{-}^{\mu}(x)=\int\frac{d^{3}k}{\sqrt{(2\pi)^{3}2\omega_{\mathbf{k}}}}\sum_{r=0}^{3}\epsilon_{r}^{*\mu}(k)a_{r}^{\dagger}(k)e^{ik\cdot x} \tag{10}$$

For the Dirac field, the Fourier decompositions are

$$\psi(x)=\int\frac{d^{3}p}{\sqrt{2(2\pi)^{3}E_{p}}}\sum_{s=1,2}\left(f_{s}(\mathbf{p})u_{s}(\mathbf{p})e^{-ip\cdot x}+\hat{f}_{s}^{\dagger}(\mathbf{p})v_{s}(\mathbf{p})e^{ip\cdot x}\right) \tag{11}$$

$$\overline{\psi}(x)=\int\frac{d^{3}p}{\sqrt{2(2\pi)^{3}E_{p}}}\sum_{s=1,2}\left(f_{s}^{\dagger}(\mathbf{p})\overline{u}_{s}(\mathbf{p})e^{ip\cdot x}+\hat{f}_{s}(\mathbf{p})\overline{v}_{s}(\mathbf{p})e^{-ip\cdot x}\right) \tag{12}$$

We can split each of these into two parts to get

$$\psi_+(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} f_s(\mathbf{p})\, u_s(\mathbf{p})\, e^{-ip\cdot x} \tag{13}$$

$$\psi_-(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \hat{f}_s^\dagger(\mathbf{p})\, v_s(\mathbf{p})\, e^{ip\cdot x} \tag{14}$$

$$\overline{\psi}_+(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} \hat{f}_s(\mathbf{p})\, \overline{v}_s(\mathbf{p})\, e^{-ip\cdot x} \tag{15}$$

$$\overline{\psi}_-(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 E_p}} \sum_{s=1,2} f_s^\dagger(\mathbf{p})\, \overline{u}_s(\mathbf{p})\, e^{ip\cdot x} \tag{16}$$

Here $\hat{f}$ is a positron operator and f is an electron operator. Thus we see that ψ_+ annihilates an electron, ψ_- creates a positron, $\overline{\psi}_+$ annihilates a positron and $\overline{\psi}_-$creates an electron.

Given all this, what interactions are possible in first order? We see from 5 that, because each of the fields can be written as the sum of two terms, when we expand out 5, we will get a sum of $2 \times 2 \times 2 = 8$ terms. Some terms are clearly impossible from the outset. For example, the term involving $\overline{\psi}_+ \psi_+ A_+$ annihilates an electron, positron and photon and creates nothing in return. Thus it corresponds to a process in which all three particles come together at the same spacetime point and mutually annihilate each other, leaving nothing. Such a process clearly cannot conserve energy or momentum, so is impossible.

Other processes might appear to be possible, however. We could, for example, envision a process in which an electron and positron are annihilated and a single photon created as a result. Such a process would be given by the term $\overline{\psi}_+ \psi_+ A_-$.

However, it turns out that *none* of the processes at first order are physically possible. We can see why by considering conservation of momentum. Suppose the two fermions have 4-momenta p and p', and the photon has 4-momentum k. Then, for any of the first order processes we must have

$$p' = \pm p \pm k \tag{17}$$

for some choice of the signs. For example, for the process $e^- + e^+ \to \gamma$ mentioned above, we must have

$$k = p + p' \tag{18}$$

Returning to 17, we can square both sides to get

$$p'^2 = p^2 + k^2 \pm 2p \cdot k \tag{19}$$

For a photon $k^2 = 0$ and, if both fermions have the same mass (as for electrons and positrons), $p'^2 = p^2 = m^2$ so we must have

$$p \cdot k = 0 \tag{20}$$

Remembering that $p \cdot k$ is the 4-dimensional scalar product, we have

$$p \cdot k = p_0 k_0 - \mathbf{p} \cdot \mathbf{k} \tag{21}$$

$$= E\omega - \mathfrak{p}\mathfrak{k}\cos\theta \tag{22}$$

where $E = p_0$ is the fermion energy, $\omega = k_0$ is the photon energy and $\mathfrak{p} \equiv |\mathbf{p}|$ and $\mathfrak{k} \equiv |\mathbf{k}|$. Therefore

$$\cos\theta = \frac{E\omega}{\mathfrak{p}\mathfrak{k}} \tag{23}$$

However, for the photon $\omega = \mathfrak{k}$ and for the fermion $E = \sqrt{\mathfrak{p}^2 + m^2}$ so we have

$$\cos\theta = \frac{\sqrt{\mathfrak{p}^2 + m^2}}{\mathfrak{p}} \tag{24}$$

$$= \sqrt{1 + \frac{m^2}{\mathfrak{p}^2}} > 1 \tag{25}$$

Thus we cannot satisfy this condition for any real angle θ, so for any first-order process, momentum cannot be conserved. Thus the lowest order at which an interaction can take place in QED is at second order.

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