ELECTRON-ELECTRON & POSITRON-POSITRON SCATTERING

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 28 Oct 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercise 9.5.

In their section 9.4, L&P give a detailed calculation of the scattering cross section for electron-electron scattering. Most of their derivation is clear so I won't go through it all again here. There are a few points where a bit more explanation might be useful, however.

To calculate the cross section, we need the square of the Feynman amplitude, which L&P give in equation 9.34:

$$\begin{aligned}|\mathscr{M}|^2 = \frac{e^2}{4}\sum_{\text{spin}}&\left\{\frac{[\overline{u}_1'\gamma_\mu u_1][\overline{u}_2'\gamma^\mu u_2]}{(p_1-p_1')^2}-\frac{[\overline{u}_1'\gamma_\mu u_2][\overline{u}_2'\gamma^\mu u_1]}{(p_1-p_2')^2}\right\}\\ &\times\left\{\frac{[\overline{u}_1\gamma_\nu u_1'][\overline{u}_2\gamma^\nu u_2']}{(p_1-p_1')^2}-\frac{[\overline{u}_2\gamma_\nu u_1'][\overline{u}_1\gamma^\nu u_2']}{(p_1-p_2')^2}\right\}\end{aligned} \tag{1}$$

Here, u_i is the Dirac spinor with four-momentum p_i and u_i' is the spinor with four-momentum p_i'. The unprimed momenta are those of the incoming electrons and the primed momenta are those of the outgoing electrons. The second term in each factor arises from swapping the outgoing electrons, and the two terms are connected by a minus sign because electrons are fermions and thus the swap is antisymmetric.

The sums over spins can be converted to traces of the various matrix products by a process similar to that which we used earlier in discussing the Dirac equation. I won't go through the derivation of all four terms in L&P's equation 9.36, but one of them is useful to examine in detail. Take T_{12} as an example. This term arises from the product of the first term in the first factor of 1 with the second term in the second factor. Looking at the numerator of this product we have

$$T_{12}=\sum_{\text{spin}}\left[\overline{u}_1'\gamma_\mu u_1\right]\left[\overline{u}_2'\gamma^\mu u_2\right]\left[\overline{u}_2\gamma_\nu u_1'\right]\left[\overline{u}_1\gamma^\nu u_2'\right] \tag{2}$$

To evaluate this, we need to write the spinor and spin indexes in full. I'll use a,b,c,d to label the spin indexes and $\alpha,\beta,\gamma,\delta,\epsilon,\theta,\zeta,\lambda$ to label the matrix components. It's helpful to write out 1 with the spin indexes first. We have

$$|\mathscr{M}|^2 = \frac{e^2}{4}\sum_{\text{spin}}\left\{\frac{\left[\overline{u}'_{1a}\gamma_\mu u_{1b}\right]\left[\overline{u}'_{2c}\gamma^\mu u_{2d}\right]}{\left(p_1-p'_1\right)^2}-\frac{\left[\overline{u}'_{1a}\gamma_\mu u_{2d}\right]\left[\overline{u}'_{2c}\gamma^\mu u_{1b}\right]}{\left(p_1-p'_2\right)^2}\right\}$$
$$\times\left\{\frac{\left[\overline{u}_{1b}\gamma_\nu u'_{1a}\right]\left[\overline{u}_{2d}\gamma^\nu u'_{2c}\right]}{\left(p_1-p'_1\right)^2}-\frac{\left[\overline{u}_{2d}\gamma_\nu u'_{1a}\right]\left[\overline{u}_{1b}\gamma^\nu u'_{2c}\right]}{\left(p_1-p'_2\right)^2}\right\} \tag{3}$$

Putting in the matrix indexes we get

All Greek indexes are summed.

$$T_{12} = \sum_{\text{spin}}\left[\overline{u}'_{1a\alpha}\left(\gamma_\mu\right)_{\alpha\beta}u_{1b\beta}\right]\left[\overline{u}'_{2c\gamma}\left(\gamma^\mu\right)_{\gamma\delta}u_{2d\delta}\right]$$
$$\times\left[\overline{u}_{2d\epsilon}\left(\gamma_\nu\right)_{\epsilon\theta}u'_{1a\theta}\right]\left[\overline{u}_{1b\zeta}\left(\gamma^\nu\right)_{\zeta\lambda}u'_{2c\lambda}\right] \tag{4}$$

There are 4 separate sums over the spins, so we can rearrange the terms to get

$$T_{12} = \sum_{a,b,c,d}\left[u'_{1a\theta}\overline{u}'_{1a\alpha}\right]\left[u_{1b\beta}\overline{u}_{1b\zeta}\right]\left[u'_{2c\lambda}\overline{u}'_{2c\gamma}\right]\left[u_{2d\delta}\overline{u}_{2d\epsilon}\right]$$
$$\times\left(\gamma_\mu\right)_{\alpha\beta}\left(\gamma^\mu\right)_{\gamma\delta}\left(\gamma_\nu\right)_{\epsilon\theta}\left(\gamma^\nu\right)_{\zeta\lambda} \tag{5}$$

We can now convert the spin sums using the earlier result

$$\sum_s u_s(\mathbf{p})\,\overline{u}_s(\mathbf{p}) = \not{p}+m \tag{6}$$

We get

$$T_{12} = \left(\not{p}'_1+m\right)_{\theta\alpha}\left(\gamma_\mu\right)_{\alpha\beta}\left(\not{p}_1+m\right)_{\beta\zeta}\left(\gamma^\nu\right)_{\zeta\lambda}\left(\not{p}'_2+m\right)_{\lambda\gamma}\left(\gamma^\mu\right)_{\gamma\delta}\left(\not{p}_2+m\right)_{\delta\epsilon}\left(\gamma_\nu\right)_{\epsilon\theta} \tag{7}$$

As all Greek indexes are summed, we see that this expression is

$$T_{12} = \text{Tr}\left[\left(\not{p}'_1+m\right)\gamma_\mu\left(\not{p}_1+m\right)\gamma^\nu\left(\not{p}'_2+m\right)\gamma^\mu\left(\not{p}_2+m\right)\gamma_\nu\right] \tag{8}$$

This is equivalent to the third of L&P's equation 9.36 since the trace of a product of matrices is cyclic. The other 3 traces in equation 9.36 are worked out in the same way.

The other equation which could use a bit of explanation is L&P's equation 9.37, which states a general trace formula:

$$\text{Tr}\left[(\not a+m)\gamma_\nu\left(\not b+m\right)\gamma_\mu\right]=4\left[a_\nu b_\mu+b_\nu a_\mu-g_{\mu\nu}\left(a\cdot b-m^2\right)\right] \tag{9}$$

We can prove this using a couple of trace formulas from L&P's appendix A. Equation A.24 tells us

$$\text{Tr}\left[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\right]=4\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}+g_{\mu\rho}+g_{\nu\lambda}\right) \tag{10}$$

Applying this, we have

$$\text{Tr}\left[\not a\gamma_\nu\not b\gamma_\mu\right]=a^\alpha b^\beta\text{Tr}\left[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right] \tag{11}$$
$$=4a^\alpha b^\beta\left(g_{\alpha\nu}g_{\beta\mu}-g_{\alpha\beta}g_{\nu\mu}+g_{\alpha\mu}+g_{\nu\beta}\right) \tag{12}$$
$$=4\left(a_\nu b_\mu-g_{\mu\nu}a\cdot b+a_\mu b_\nu\right) \tag{13}$$

To evaluate the other traces on the LHS of 9 we use the result that the trace of the product of an odd number of γ^μ is zero (again, L&P appendix A). Thus we're left with

$$\text{Tr}\left[m\gamma_\nu m\gamma_\mu\right]=m^2\text{Tr}\left[\gamma_\nu\gamma_\mu\right] \tag{14}$$
$$=4m^2g_{\mu\nu} \tag{15}$$

using L&P equation A.23. Combining these results gives us 9.

The remainder of the derivation is fairly clear in L&P, although somewhat lengthy. The final expression (in the centre of mass frame) for $|\mathscr{M}|^2$ as a function of the energy E, magnitude $\mathfrak{p}$ of the 3-momentum and scattering angle θ is

In the centre of mass frame, E and $\mathfrak{p}$ are the same for all 4 electrons.

$$|\mathscr{M}|^2=\frac{2e^4}{\mathfrak{p}^4}\left\{\frac{\left(E^2+\mathfrak{p}^2\right)^2+\left(E^2+\mathfrak{p}^2\cos\theta\right)^2-2m^2\mathfrak{p}^2\left(1-\cos\theta\right)}{\left(1-\cos\theta\right)^2}\right.$$
$$+\frac{\left(E^2+\mathfrak{p}^2\right)^2+\left(E^2-\mathfrak{p}^2\cos\theta\right)^2-2m^2\mathfrak{p}^2\left(1+\cos\theta\right)}{\left(1+\cos\theta\right)^2}$$
$$\left.+2\,\frac{\left(E^2+\mathfrak{p}^2\right)^2-2m^2\left(E^2+\mathfrak{p}^2\right)}{\sin^2\theta}\right\} \tag{16}$$

To work out the scattering cross section for positron-positron scattering, we can trace the derivation through with electrons replaced by positrons. This means that in 1 we replace all u spinors by v spinors. When working out the traces in L&P's equation 9.36, we now use the earlier result

$$\sum_s v_s(\mathbf{p})\,\overline{v}_s(\mathbf{p}) = \not{p} - m \tag{17}$$

The only difference is that all $+m$ terms in L&P's equations 9.36 are replaced by $-m$. If we trace through L&P's evaluation of these traces in their equations 9.37 through 9.47, we see that the mass enters only as even powers (m^2 and m^4). Thus the sign of m cancels out and we end up with the same result 16.

PINGBACKS

Pingback: Electron-electron scattering: High-energy limit