ELECTRON-ELECTRON SCATTERING: HIGH-ENERGY LIMIT

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercise 9.6.

The Feynman amplitude for electron-electron scattering is

$$\begin{aligned} |\mathscr{M}|^2 = \frac{2e^4}{\mathfrak{p}^4}\Bigg\{ & \frac{\left(E^2+\mathfrak{p}^2\right)^2+\left(E^2+\mathfrak{p}^2\cos\theta\right)^2-2m^2\mathfrak{p}^2\left(1-\cos\theta\right)}{\left(1-\cos\theta\right)^2} \\ & +\frac{\left(E^2+\mathfrak{p}^2\right)^2+\left(E^2-\mathfrak{p}^2\cos\theta\right)^2-2m^2\mathfrak{p}^2\left(1+\cos\theta\right)}{\left(1+\cos\theta\right)^2} \\ & +2\,\frac{\left(E^2+\mathfrak{p}^2\right)^2-2m^2\left(E^2+\mathfrak{p}^2\right)}{\sin^2\theta}\Bigg\} \end{aligned} \tag{1}$$

For high energies, $E\gg m$ and $E\approx\mathfrak{p}$ so we can approximate this by

$$|\mathscr{M}|^2\to\frac{2e^4}{E^4}E^4\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{4+(1-\cos\theta)^2}{(1+\cos\theta)^2}+\frac{8}{\sin^2\theta}\right] \tag{2}$$

We can simplify this using the trig half-angle formulas:

$$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2} \tag{3}$$

$$\cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2} \tag{4}$$

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{5}$$

To simplify the notation, we'll define

$$s\equiv\sin\frac{\theta}{2} \tag{6}$$

$$c\equiv\cos\frac{\theta}{2} \tag{7}$$

with

$$s^2+c^2=1 \tag{8}$$

We get from 2

$$\frac{|\mathscr{M}|^2}{2e^4} \to \left[\frac{1+c^4}{s^4}+\frac{1+s^4}{c^4}+\frac{2}{s^2c^2}\right] \tag{9}$$

$$= \frac{1+\left(1-s^2\right)^2}{s^4}+\frac{1+\left(1-c^2\right)^2}{c^4}+\frac{2}{s^2c^2} \tag{10}$$

$$= \frac{2-2s^2+s^4}{s^4}+\frac{2-2c^2+c^4}{c^4}+\frac{2}{s^2c^2} \tag{11}$$

$$= 2\left(\frac{1}{s^4}+\frac{1}{c^4}+1-\frac{c^2+s^2}{s^2c^2}+\frac{1}{s^2c^2}\right) \tag{12}$$

$$= 2\left(\frac{1}{s^4}+\frac{1}{c^4}+1\right) \tag{13}$$

$$|\mathscr{M}|^2 = 4e^4\left(\frac{1}{\sin^4\frac{\theta}{2}}+\frac{1}{\cos^4\frac{\theta}{2}}+1\right) \tag{14}$$

Inserting this into L&P's equation 9.44 for the differential cross section (using L&P's $s=4E^2$) we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 4E^2}|\mathscr{M}|^2 \tag{15}$$

$$= \frac{e^4}{16\pi^2 4E^2}\left(\frac{1}{\sin^4\frac{\theta}{2}}+\frac{1}{\cos^4\frac{\theta}{2}}+1\right) \tag{16}$$

$$= \frac{\alpha^2}{4E^2}\left(\frac{1}{\sin^4\frac{\theta}{2}}+\frac{1}{\cos^4\frac{\theta}{2}}+1\right) \tag{17}$$

where

$$\alpha \equiv \frac{e^2}{4\pi} \tag{18}$$

is the fine-structure constant.