COMPTON SCATTERING: CROSS-SECTION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercise 9.7.

Note: This isn't really a solution of exercise 9.7; rather it's a summary of the derivation of equation 9.111.

In section 9.8, L&P give a detailed derivation of the scattering cross-section for Compton scattering (elastic photon-electron scattering). Here, I will fill in a few of the steps. The reader should refer to the textbook for a complete derivation, as it is far too lengthy to repeat here.

The first point is that we can shift the polarization vectors according to L&P's equation 9.68:

$$\epsilon_\mu(k) \to \epsilon_\mu(k) + k_\mu \vartheta \tag{1}$$

where k is the 4-momentum of the photon and ϑ is an aribtrary constant. Making this substitution does not affect the value of the S-matrix elements. This is a form of gauge invariance. We can use this condition to set one of the components of ϵ_μ to zero. If we choose, for some initial polarization vector ϵ:

$$\vartheta = -\frac{\epsilon_0(k)}{k_0} \tag{2}$$

for example, then the transformed polarization vector is

$$\epsilon^\mu = (0, \epsilon) \tag{3}$$

where ϵ is the 3-vector giving the spatial components of ϵ^μ. This choice is made by L&P in equation 9.79. Using this choice, they grind through the calculation of the traces of matrices required to get the cross-section, with the result given in equation 9.107:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\epsilon\cdot\epsilon'\right)^2 - 2\right] \tag{4}$$

where α is the fine structure constant, ω is the energy of the incident photon, ω' is the energy of the outgoing photon and $\epsilon = \epsilon_\mu(k)$ and $\epsilon' = \epsilon'_\mu(k')$. The outgoing photon energy is related to the incident energy by equation 7.115, which gives

$$\omega' = \frac{\omega}{1 + (\omega/m)(1 - \cos\theta)} \tag{5}$$

where θ is the scattering angle.

This result is for a particular set of incident and outgoing photon polarizations. If we wish to determine only the cross-section for scattering regardless of the polarizations of either photon, we need to sum over the incident and final polarizations and then take the average. Since photons are transversely polarized, there are two independent polarization states. We therefore need to sum over incident and final polarizations and then divide by 2 to average over the final polarizations. The only term in 4 which depends on polarization is the one involving $\epsilon \cdot \epsilon'$. For all other terms, the effect of this summing and averaging is thus to multiply by 4 (for the 2 possible polarizations in each of the incident and final photons) and then divide by 2, giving a net factor of 2. We therefore have

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[2\frac{\omega'}{\omega} + 2\frac{\omega}{\omega'} + 2\sum_{\text{pol}}\left(\epsilon \cdot \epsilon'\right)^2 - 4\right] \tag{6}$$

Here, we have multiplied all terms within the brackets except the one involving $\epsilon \cdot \epsilon'$ by 2, and summed the one involving $\epsilon \cdot \epsilon'$ over polarizations and then divided this term by 2.

To do this sum, we need equation 9.109 which appears seemingly from nowhere:

$$\sum_r \epsilon_{ri}(k)\,\epsilon_{rj}(k) = \delta_{ij} - \frac{k_i k_j}{\mathfrak{k}^2} \tag{7}$$

where the subscript r refers to the polarization, i and j refer to the spatial component of the vector and $\mathfrak{k} = |\mathbf{k}|$ and we're taking ϵ to be real. There are a couple of ways to see where it comes from.

First, we can remember that ϵ_1 and ϵ_2 are the transverse polarization vectors, so they are perpendicular to each other and also to the 3-momentum **k**, so they form a basis in 3-d space. We can take the dot product of each of these basis vectors with 7, as follows (where now a repeated i indicates a sum over $i = 1, 2, 3$):

$$\epsilon_{1i}\sum_r \epsilon_{ri}(k)\,\epsilon_{rj}(k) = \delta_{ij}\epsilon_{1i} - \frac{\epsilon_{1i} k_i k_j}{\mathfrak{k}^2} \tag{8}$$

We now have

$$\epsilon_{1i}\epsilon_{2i} = 0 \tag{9}$$

$$\epsilon_{ri}\epsilon_{ri} = 1 \tag{10}$$

$$\epsilon_{1i}k_i = 0 \tag{11}$$

because the polarization vectors are unit vectors and ϵ_1 is perpendicular to both ϵ_2 and **k**. Therefore, 8 becomes

$$\epsilon_{1i}\sum_r \epsilon_{ri}(k)\,\epsilon_{rj}(k) = \epsilon_{1i}\epsilon_{1i}\epsilon_{1j} = \epsilon_{1j} \tag{12}$$

$$\delta_{ij}\epsilon_{1i} - \frac{\epsilon_{1i}k_ik_j}{\mathbf{k}^2} = \epsilon_{1j} - 0 = \epsilon_{1j} \tag{13}$$

Thus the two sides of 8 are equal. A similar argument applies to multiplying by ϵ_{2i}. Finally, we can multiply by k_i to get

$$\sum_r k_i\epsilon_{ri}(k)\,\epsilon_{rj}(k) = k_i\delta_{ij} - \frac{k_ik_ik_j}{\mathbf{k}^2} \tag{14}$$

The LHS is zero because of 11 and the RHS is

$$k_i\delta_{ij} - \frac{k_ik_ik_j}{\mathbf{k}^2} = k_j - \frac{\mathbf{k}^2k_j}{\mathbf{k}^2} = 0 \tag{15}$$

Thus again, the two sides are equal. Any two vectors that have the same scalar product with all three basis vectors are equal, so 7 is true.

Another way of validating 7 is to go back to equation 8.87 and set $k^2 = 0$ for a photon. After inserting 8.79 into this, we get

$$\sum_{r=1,2}\epsilon_r^\mu(k)\,\epsilon_r^\nu(k) = -g^{\mu\nu} - \frac{k^\mu k^\nu - (k\cdot n)(n^\mu k^\nu + k^\mu n^\nu)}{(k\cdot n)^2} \tag{16}$$

Using the definition of $n^\mu = (1,0,0,0)$ from equation 8.76 and restricting μ and ν to the spatial components i and j, we again get 7.

Reverting back to the cross-section formula, we now have from equation 9.108:

$$\sum_{r,r'}\left[\epsilon_r(k)\cdot\epsilon_{r'}\left(k'\right)\right]^2=\left[\sum_r\epsilon_{ri}(k)\cdot\epsilon_{rj}(k)\right]\left[\sum_{r'}\epsilon_{r'i}\left(k'\right)\cdot\epsilon_{r'j}\left(k'\right)\right] \tag{17}$$

$$=\left[\delta_{ij}-\frac{k_ik_j}{\mathfrak{k}^2}\right]\left[\delta_{ij}-\frac{k'_ik'_j}{\mathfrak{k}'^2}\right] \tag{18}$$

$$=\delta_{ij}\delta_{ij}+\frac{k_ik_j}{\mathfrak{k}^2}\frac{k'_ik'_j}{\mathfrak{k}'^2}-\frac{k_ik_i}{\mathfrak{k}^2}-\frac{k'_ik'_i}{\mathfrak{k}'^2} \tag{19}$$

$$=3+\frac{\left(\mathbf{k}\cdot\mathbf{k}'\right)^2}{\left(\mathfrak{k}\mathfrak{k}'\right)^2}-1-1 \tag{20}$$

$$=1+\cos^2\theta \tag{21}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Inserting this into 6 we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}}=\frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[2\frac{\omega'}{\omega}+2\frac{\omega}{\omega'}+2\left(1+\cos^2\theta\right)-4\right] \tag{22}$$

$$=\frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+\cos^2\theta-1\right] \tag{23}$$

$$=\frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right] \tag{24}$$

Exercise 9.7 asks us to find this formula by using the sum given in equation 8.88, which is

$$\sum_{r=1,2}\epsilon_r^{\mu}(k)\,\epsilon_r^{\nu}(k)=-g^{\mu\nu} \tag{25}$$

I have to confess I don't understand how to justify this formula, nor how we can use it to derive the formula 24. Any ideas are welcome - please use the comment link at the top of this post.

PINGBACKS

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