QUANTUM ELECTRODYNAMICS: SCATTERING BY A HEAVY NUCLEUS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 9, Exercise 9.8.

We've seen that in quantum electrodynamics, no first-order scattering processes are possible, since we cannot conserve 4-momentum in such processes. However, in the case where a fermion (such as an electron) is scattered by a heavy nucleus, or by some static external electric field, we can get an approximation for the scattering amplitude by neglecting the momentum transferred from the scattered particle to the nucleus. In other words, we assume that the nucleus remains at rest (in the lab frame) throughout the scattering process. This is done by splitting the electric vector potential into two parts: one part $A_{e\mu}$ represents the external field (with the subscript e standing for 'external') and the other part A_μ being the usual vector potential with which we have dealt in earlier posts. This means we can write the interaction term in the Lagrangian as

$$\mathscr{L}_{\text{int}} = e :\overline{\psi}\left(\not{A} + \not{A}_e\right)\psi: \tag{1}$$

The term involving $\not{A}$ still conserves 4-momentum and thus cannot appear in any first-order S-matrix element. The external part, however, does have a first-order component, and this is derived in L&P's section 9.9, with equation 9.118:

$$\left\langle e^-\left(p'\right)\left|S_e^{(1)}\right|e^-\left(p\right)\right\rangle = \frac{2\pi\delta\left(E-E'\right)}{\sqrt{2EV}\sqrt{2E'V}} i\mathscr{M}_{fi} \tag{2}$$

with

$$\mathscr{M}_{fi} = e\overline{u}\left(p'\right)\not{A}_e\left(\mathbf{p}'-\mathbf{p}\right)u\left(p\right) \tag{3}$$

where primed symbols refer to the outgoing electron and unprimed symbols to the incoming electron.

From the $\delta\left(E-E'\right)$ in 2, we see that the energy of the incoming electron is equal to the energy of the scattered electron. However, the absence of a delta function for the 3-momentum means that this is not conserved. The magnitude $\mathfrak{p}=|\mathbf{p}|$ *is* conserved, since the energy satisfies $E=\sqrt{\mathfrak{p}^2+m^2}$, but the directions of $\mathbf{p}'$ and $\mathbf{p}$ could be different. As this result is obtained from the first-order S-matrix element alone, it is only an approximation.

[In L&P's exercise 9.8, they ask us to show that $E\approx E'$. This appears obvious because of the $\delta\left(E-E'\right)$ term in the S-matrix element, so I don't know if I'm missing something deeper.]

In any event, they proceed to calculate the differential scattering cross-section for this process, which is

$$d\sigma=\frac{1}{2\mathfrak{p}}\frac{d^3p'}{\left(2\pi\right)^3 2E'}\left(2\pi\right)\delta\left(E-E'\right)\overline{|\mathscr{M}|^2} \tag{4}$$

where $\mathfrak{p}$ is the magnitude of the 3-momentum of the incident electron, which is

$$\mathfrak{p}=\sqrt{E^2-m^2} \tag{5}$$

This is the element of cross-section for scattering into the element d^3p' in momentum space. To get the differential cross-section for scattering into an element of solid angle $d\Omega$, we must integrate 4 over all values of p' (but not over angle). That is, we want

$$\frac{d\sigma}{d\Omega}=\frac{\overline{|\mathscr{M}|^2}}{16\pi^2\mathfrak{p}}\int_0^\infty\frac{\delta\left(E-E'\right)}{E'}p'^2dp' \tag{6}$$

To do this integral, we need to convert the delta function so that its argument is p', which we do using the formula

$$\delta\left(f\left(x\right)\right)=\frac{\delta\left(x-x_0\right)}{\left|f'\left(x_0\right)\right|} \tag{7}$$

where x_0 is the only zero of $f\left(x\right)$. In our case we have

$$E-E'=E-\sqrt{p'^2+m^2} \tag{8}$$

$$\frac{d\left(E-E'\right)}{dp'}=-\frac{p'}{\sqrt{p'^2+m^2}} \tag{9}$$

$$p'_0=\sqrt{E^2-m^2} \tag{10}$$

$$\left.\frac{d\left(E-E'\right)}{dp'}\right|_{p'_0}=-\frac{\sqrt{E^2-m^2}}{E} \tag{11}$$

so we have

$$\delta\left(E-E'\right)=\frac{E\delta\left(p'-\sqrt{E^2-m^2}\right)}{\sqrt{E^2-m^2}} \tag{12}$$

Putting this and 5 into 6 we have (using $E'=E$ because of energy conservation):

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\overline{|\mathscr{M}|^2}}{16\pi^2\sqrt{E^2-m^2}}\int_0^\infty \frac{E\delta\left(p'-\sqrt{E^2-m^2}\right)}{E\sqrt{E^2-m^2}}p'^2dp' && (13)\\
&= \frac{\overline{|\mathscr{M}|^2}}{16\pi^2}\frac{\left(\sqrt{E^2-m^2}\right)^2}{\sqrt{E^2-m^2}\sqrt{E^2-m^2}} && (14)\\
&= \frac{\overline{|\mathscr{M}|^2}}{16\pi^2} && (15)
\end{aligned}$$

The remainder of the derivation is given in the book, leading to the final formula, equation 9.131:

$$\frac{d\sigma}{d\Omega}=\frac{(Z\alpha)^2}{4E^2v^4\sin^4\frac{\theta}{2}}\left(1-v^2\sin^2\frac{\theta}{2}\right) \tag{16}$$

where Z is the charge on the nucleus, α is the fine structure constant, v is the speed of the incident electron and θ is the scattering angle (the angle between $\mathbf{p}$ and $\mathbf{p}'$). In the non-relativistic limit, $v \ll 1$ and we get

$$\frac{d\sigma}{d\Omega}\approx\frac{(Z\alpha)^2}{4E^2v^4\sin^4\frac{\theta}{2}} \tag{17}$$

which is the Rutherford scattering formula.