PARITY TRANSFORMATION OF CREATION AND ANNIHILATION OPERATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 27 Nov 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.2.

We've seen that parity can be treated as a transformation of the operators used in quantum mechanics, so that an operator Ω transforms under the parity operator $\mathcal{P}$ according to

$$\Omega \to \mathcal{P}\Omega\mathcal{P}^{-1} \tag{1}$$

In quantum field theory, the fields are the operators, so we'd expect to apply parity in a similar manner to these field operators. In section 10.2, L&P thus propose that for a scalar field $\phi(x)$, the parity transformation behaves as follows

$$\phi_P(x) \equiv \mathcal{P}\phi(x)\mathcal{P}^{-1} = \eta_P\phi(\tilde{x}) \tag{2}$$

where $\phi_P(x)$ is the transformed field written in terms of the original space-time coordinates x, and $\phi(\tilde{x})$ is the original field written in terms of the parity-transformed coordinates

$$\tilde{x} = (t, -\mathbf{x}) \tag{3}$$

and η_P is a constant (independent of spacetime).

L&P show in their equations 10.9 and 10.10 that if the scalar field Lagrangian is to be invariant under a parity transformation, then

$$\eta_P = \pm 1 \tag{4}$$

since all terms in the Lagrangian involve the square of the field or of its derivatives. The factor η_P is called the *intrinsic parity*, and for a free scalar particle, both values of η_P are possible.

We can now see how the creation and annihilation operators for the scalar field transform under parity. From our earlier calculations, we can write these operators in terms of the field operators:

$$a(p) = \frac{1}{(2\pi)^{3/2}} \int d^3x\, e^{ip\cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(x) + \frac{i}{\sqrt{2E_p}} \Pi(x) \right) \quad (5)$$

$$a^\dagger(p) = \frac{1}{(2\pi)^{3/2}} \int d^3x\, e^{-ip\cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(x) - \frac{i}{\sqrt{2E_p}} \Pi(x) \right) \quad (6)$$

where

$$\Pi(x) = \dot{\phi}(x) \quad (7)$$

The parity transformation of $a(p)$ gives

$$\mathcal{P}a(p)\mathcal{P}^{-1} = \frac{1}{(2\pi)^{3/2}} \int d^3x\, e^{ip\cdot x} \left(\sqrt{\frac{E_p}{2}} \mathcal{P}\phi(x)\mathcal{P}^{-1} + \frac{i}{\sqrt{2E_p}} \mathcal{P}\dot{\phi}(x)\mathcal{P}^{-1} \right) \quad (8)$$

Here, we're assuming that $\mathcal{P}$ affects only operators, so since the other factors on the RHS are all ordinary numbers, we're justified in taking $\mathcal{P}\ldots\mathcal{P}^{-1}$ operation inside the integral and applying it to the field operators.

Using 2 and 3, we have

$$\mathcal{P}a(p)\mathcal{P}^{-1} = \frac{\eta_P}{(2\pi)^{3/2}} \int d^3x\, e^{ip\cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(\tilde{x}) + \frac{i}{\sqrt{2E_p}} \dot{\phi}(\tilde{x}) \right) \quad (9)$$

$$= \frac{\eta_P}{(2\pi)^{3/2}} \int d^3x\, e^{ip\cdot x} \left(\sqrt{\frac{E_p}{2}} \phi(t,-\mathbf{x}) + \frac{i}{\sqrt{2E_p}} \dot{\phi}(t,-\mathbf{x}) \right) \quad (10)$$

We can now change the variable of integration from $x = (t,\mathbf{x})$ to $x' = (t,-\mathbf{x})$. This changes the sign of d^3x but also reverses the direction of integration for all 3 components of **x**, which also reverses the sign, so the two sign reversals cancel each other and we're left with

$$\mathcal{P}a(p)\mathcal{P}^{-1} = \frac{\eta_P}{(2\pi)^{3/2}} \int d^3x'\, e^{ip\cdot x'} \left(\sqrt{\frac{E_p}{2}} \phi(t,\mathbf{x}) + \frac{i}{\sqrt{2E_p}} \dot{\phi}(t,\mathbf{x}) \right) \quad (11)$$

Since

$$p\cdot x' = E_{\mathbf{p}}t - \mathbf{p}\cdot(-\mathbf{x}) \quad (12)$$

$$= E_{\mathbf{p}}t + \mathbf{p}\cdot\mathbf{x} \quad (13)$$

where $E_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$, we have

$$\mathcal{P}a(p)\mathcal{P}^{-1}=\frac{\eta_P}{(2\pi)^{3/2}}\int d^3x\, e^{iE_pt}e^{i\mathbf{p}\cdot\mathbf{x}}\left(\sqrt{\frac{E_p}{2}}\phi(t,\mathbf{x})+\frac{i}{\sqrt{2E_p}}\dot{\phi}(t,\mathbf{x})\right) \tag{14}$$

Since $E_{\mathbf{p}}=E_{-\mathbf{p}}$, this is the same as 5 with $\mathbf{p}$ replaced by $-\mathbf{p}$. In other words, parity reverses the direction of momentum in the annihilation operator, without changing the energy. We can write this as

$$\mathcal{P}a(p)\mathcal{P}^{-1}=\mathcal{P}a(E_{\mathbf{p}},\mathbf{p})\mathcal{P}^{-1}=\eta_Pa(E_{\mathbf{p}},-\mathbf{p}) \tag{15}$$

The same argument applies to the creation operator, so we also have

$$\mathcal{P}a^{\dagger}(p)\mathcal{P}^{-1}=\mathcal{P}a^{\dagger}(E_{\mathbf{p}},\mathbf{p})\mathcal{P}^{-1}=\eta_Pa^{\dagger}(E_{\mathbf{p}},-\mathbf{p}) \tag{16}$$

If we assume that parity has no effect on the vacuum state so that $\mathcal{P}|0\rangle=|0\rangle$, then we have

$$\mathcal{P}|\mathbf{k}\rangle=\mathcal{P}a^{\dagger}(E_{\mathbf{k}},\mathbf{k})|0\rangle \tag{17}$$

$$=\mathcal{P}a^{\dagger}(E_{\mathbf{k}},\mathbf{k})\mathcal{P}^{-1}\mathcal{P}|0\rangle \tag{18}$$

$$=\eta_Pa^{\dagger}(E_{\mathbf{k}},-\mathbf{k})|0\rangle \tag{19}$$

$$=\eta_P|-\mathbf{k}\rangle \tag{20}$$

PINGBACKS

Pingback: Maxwell's equations for free fields are invariant under parity