# PARITY TRANSFORMATION OF SINGLE PARTICLE AND ANTIPARTICLE STATES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercises 10.3-10.4.

In their section 10.2.2, L&P show that applying the parity operator $\mathcal{P}$ to the Dirac field vector $\psi(x)$ yields

$$\mathcal{P}\psi(x)\mathcal{P}^{-1}=\eta_P\gamma_0\psi(\tilde{x}) \tag{1}$$

where $\eta_P=\pm1$ and

$$\tilde{x}=(t,-\mathbf{x}) \tag{2}$$

We can use this result to determine the effect of a parity transformation on a single particle or antiparticle fermion state.

First, we look at the Dirac-Pauli representation of $\gamma_0$ and the spinors $u_s$ and $v_s$:

$$\gamma_0=\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix} \tag{3}$$

$$u_\pm(\mathbf{p})=\sqrt{E_p+m}\begin{bmatrix} \chi_\pm\\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\pm\end{bmatrix} \tag{4}$$

$$v_\pm(\mathbf{p})\pm\sqrt{E_p+m}\begin{bmatrix} \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\mp\\ \chi_\mp\end{bmatrix} \tag{5}$$

where $I$ is the $2\times2$ identity matrix and

$$\chi_+=\begin{bmatrix} 1\\ 0\end{bmatrix} \tag{6}$$

$$\chi_-=\begin{bmatrix} 0\\ 1\end{bmatrix} \tag{7}$$

and $\sigma$ is the vector consisting of the three Pauli matrices. From these, we see that (since $E_{-\mathbf{p}}=E_{\mathbf{p}}$)

$$\gamma_0 u_s(\mathbf{p}) = \sqrt{E_p+m}\begin{bmatrix}\chi_\pm \\ -\frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\pm\end{bmatrix} \tag{8}$$

$$= \sqrt{E_p+m}\begin{bmatrix}\chi_\pm \\ \frac{\sigma\cdot(-\mathbf{p})}{E_p+m}\chi_\pm\end{bmatrix} \tag{9}$$

$$= u_s(-\mathbf{p}) \tag{10}$$

and

L&P state that these results must be valid in any representation of the Dirac matrices, though I'm not sure why.

$$\gamma_0 v_s(\mathbf{p}) = \sqrt{E_p+m}\begin{bmatrix}\frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\mp \\ -\chi_\mp\end{bmatrix} \tag{11}$$

$$= -\sqrt{E_p+m}\begin{bmatrix}\frac{\sigma\cdot(-\mathbf{p})}{E_p+m}\chi_\mp \\ \chi_\mp\end{bmatrix} \tag{12}$$

$$= -v_s(-\mathbf{p}) \tag{13}$$

To apply these results to the problem of determining the behaviour of single particle and antiparticle states under parity, we need the forms of the corresponding creation operators in terms of the field operators, which are

$$f_s^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, \psi^\dagger(x)\, u_s(\mathbf{p})\, e^{-ip\cdot x} \tag{14}$$

$$\hat{f}_s^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\, v_s^\dagger(\mathbf{p})\, \psi(x)\, e^{-ip\cdot x} \tag{15}$$

From 1, using the facts that $\mathcal{P}$ is unitary, so $\mathcal{P}^{-1} = \mathcal{P}^\dagger$ and $\gamma_0^\dagger = \gamma_0$, we have, for the LHS:

$$\left(\mathcal{P}\psi(x)\mathcal{P}^{-1}\right)^\dagger = \left(\mathcal{P}^{-1}\right)^\dagger \psi^\dagger(x)\mathcal{P}^\dagger \tag{16}$$

$$= \mathcal{P}\psi^\dagger(x)\mathcal{P}^{-1} \tag{17}$$

and for the RHS:

$$\left(\eta_P\gamma_0\psi(\tilde{x})\right)^\dagger = \eta_P\psi^\dagger(\tilde{x})\gamma_0 \tag{18}$$

so

$$\mathcal{P}\psi^\dagger(x)\mathcal{P}^{-1} = \eta_P\psi^\dagger(\tilde{x})\gamma_0 \tag{19}$$

Therefore, from 14 and 10

$$\mathcal{P}f_s^{\dagger}(\mathbf{p})\mathcal{P}^{-1} = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\,\mathcal{P}\psi^{\dagger}(x)\mathcal{P}^{-1}u_s(\mathbf{p})e^{-ip\cdot x} \tag{20}$$

$$= \frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\,\eta_P\psi^{\dagger}(\tilde{x})\gamma_0 u_s(\mathbf{p})e^{-ip\cdot x} \tag{21}$$

$$= \frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\,\eta_P\psi^{\dagger}(\tilde{x})u_s(-\mathbf{p})e^{-ip\cdot x} \tag{22}$$

We can now change the integration variable from $\mathbf{x}$ to $-\mathbf{x}$, which will replace the $\tilde{x}$ by $x$ in the field operator and give us

$$\mathcal{P}f_s^{\dagger}(\mathbf{p})\mathcal{P}^{-1} = \frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\,\eta_P\psi^{\dagger}(x)u_s(-\mathbf{p})e^{-iE_{\mathbf{p}}t}e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{23}$$

Since $E_{-\mathbf{p}} = E_{\mathbf{p}}$, this is equivalent to

$$\mathcal{P}f_s^{\dagger}(\mathbf{p})\mathcal{P}^{-1} = \eta_P f_s^{\dagger}(-\mathbf{p}) \tag{24}$$

The action of $\mathcal{P}$ on a single particle state is then (assuming parity has no effect on the vacuum state):

$$\mathcal{P}|\mathbf{p}\rangle = \mathcal{P}f_s^{\dagger}(\mathbf{p})|0\rangle \tag{25}$$

$$= \mathcal{P}f_s^{\dagger}(\mathbf{p})\mathcal{P}^{-1}\mathcal{P}|0\rangle \tag{26}$$

$$= \eta_P f_s^{\dagger}(-\mathbf{p})|0\rangle \tag{27}$$

$$= \eta_P|-\mathbf{p}\rangle \tag{28}$$

For an antiparticle we start from 15 and use 13, we get

$$\mathcal{P}\hat{f}_s^{\dagger}(\mathbf{p})\mathcal{P}^{-1} = \frac{1}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\ v_s^{\dagger}(\mathbf{p})\mathcal{P}\psi(x)\mathcal{P}^{-1}e^{-ip\cdot x} \tag{29}$$

$$= \frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\ v_s^{\dagger}(\mathbf{p})\gamma_0\psi(\tilde{x})e^{-ip\cdot x} \tag{30}$$

$$= \frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\ \left(\gamma_0 v_s(\mathbf{p})\right)^{\dagger}\psi(\tilde{x})e^{-ip\cdot x} \tag{31}$$

$$= -\frac{\eta_P}{\sqrt{2(2\pi)^3 E_p}}\int d^3x\ v_s^{\dagger}(-\mathbf{p})\psi(\tilde{x})e^{-ip\cdot x} \tag{32}$$

$$= -\eta_P\hat{f}_s^{\dagger}(-\mathbf{p}) \tag{33}$$

Therefore, for an antiparticle with momentum **k** we have

$$\mathcal{P}|\mathbf{k}\rangle = \mathcal{P}\hat{f}_s^{\dagger}(\mathbf{k})|0\rangle \tag{34}$$

$$= \mathcal{P}\hat{f}_s^{\dagger}(\mathbf{k})\mathcal{P}^{-1}\mathcal{P}|0\rangle \tag{35}$$

$$= -\eta_P\hat{f}_s^{\dagger}(-\mathbf{k})|0\rangle \tag{36}$$

$$= -\eta_P|-\mathbf{k}\rangle \tag{37}$$

Thus the single particle and antiparticle states transform oppositely under parity.