MAXWELL'S EQUATIONS FOR FREE FIELDS ARE INVARIANT UNDER PARITY

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.5.

In their section 10.2.3, L&P show that in order for the Lagrangian for the free electromagnetic field to be invariant under parity, the field operators must satisfy one of two conditions. Either

$$A_P^0(x) \equiv \mathcal{P}A^0(x)\mathcal{P}^{-1} = A^0(\tilde{x}) \tag{1}$$

$$\mathbf{A}_P(x) \equiv \mathcal{P}\mathbf{A}(x)\mathcal{P}^{-1} = -\mathbf{A}(\tilde{x}) \tag{2}$$

or

$$A_P^0(x) \equiv \mathcal{P}A^0(x)\mathcal{P}^{-1} = -A^0(\tilde{x}) \tag{3}$$

$$\mathbf{A}_P(x) \equiv \mathcal{P}\mathbf{A}(x)\mathcal{P}^{-1} = \mathbf{A}(\tilde{x}) \tag{4}$$

where

$$\tilde{x} \equiv (t, -\mathbf{x}) \tag{5}$$

For free fields, that is, in the absence of charges serving as sources of the fields, there is no way to distinguish between these two options. In fact, both conditions leave Maxwell's equations for free fields invariant under parity. To see this, we write Maxwell's equations in terms of the field operators (which are the quantum field versions of the potentials ϕ and $\mathbf{A}$ in classical electromagnetism). The equations in their tradition forms in terms of electric and magnetic fields are

$$\nabla \cdot \mathbf{E} = \rho \tag{6}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{9}$$

E and **B** are written in terms of the field operators as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{10}$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t} \tag{11}$$

The first Maxwell equation 6 for free fields ($\rho = 0$) is therefore

$$-\nabla_{\mathbf{x}}^2 A^0(x) - \frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}} \cdot \mathbf{A}(x)\right) = 0 \tag{12}$$

where the suffix **x** on the ∇ operators indicates derivatives with respect to **x**, the untransformed coordinates.

Now suppose we apply a parity transformation and use the first condition 1 for transforming the fields. This gives us

$$-\nabla_{\mathbf{x}}^2 A^0(\tilde{x}) - \frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}} \cdot (-\mathbf{A}(\tilde{x}))\right) = 0 \tag{13}$$

$$-\nabla_{\mathbf{x}}^2 A^0(\tilde{x}) + \frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}} \cdot \mathbf{A}(\tilde{x})\right) = 0 \tag{14}$$

Since parity inverts the spatial components of x, we can write this as

$$-\nabla_{-\mathbf{x}}^2 A^0(\tilde{x}) - \frac{\partial}{\partial t}\left(\nabla_{-\mathbf{x}} \cdot \mathbf{A}(\tilde{x})\right) = 0 \tag{15}$$

Note that the sign of the second term has changed since it contains a first derivative $\nabla_{-\mathbf{x}}$. The sign of the first term remains unchanged under parity since it contains a second derivative $(-\nabla_{-\mathbf{x}}) \cdot (-\nabla_{-\mathbf{x}}) = +\nabla_{-\mathbf{x}}^2$. Thus the transformed equation is the same as the original 12 in which **x** has been swapped with $-\mathbf{x}$.

If we use the second condition 3, we get after a parity transformation:

$$-\nabla_{\mathbf{x}}^{2}\left(-A^{0}\left(\tilde{x}\right)\right)-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\cdot\mathbf{A}\left(\tilde{x}\right)\right)=0 \tag{16}$$

$$+\nabla_{\mathbf{x}}^{2}A^{0}\left(\tilde{x}\right)-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\cdot\mathbf{A}\left(\tilde{x}\right)\right)=0 \tag{17}$$

$$\nabla_{-\mathbf{x}}^{2}A^{0}\left(\tilde{x}\right)+\frac{\partial}{\partial t}\left(\nabla_{-\mathbf{x}}\cdot\mathbf{A}\left(\tilde{x}\right)\right)=0 \tag{18}$$

If we multiply the last line through by -1, we regain 15, so that 6 remains invariant under parity using 3. Note that this last step is valid only if the RHS is zero, that is, if there are no sources.

The other 3 equations can be verified similarly. For 7, we have, applying 10 and 11:

$$\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\mathbf{A}\left(x\right)+\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}A^{0}\left(x\right)\right)+\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(x\right)=0 \tag{19}$$

Transforming using 1, we have

$$-\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\mathbf{A}\left(\tilde{x}\right)+\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}A^{0}\left(\tilde{x}\right)\right)-\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(\tilde{x}\right)=0 \tag{20}$$

$$-\nabla_{-\mathbf{x}}\times\nabla_{-\mathbf{x}}\times\mathbf{A}\left(\tilde{x}\right)-\frac{\partial}{\partial t}\left(\nabla_{-\mathbf{x}}A^{0}\left(\tilde{x}\right)\right)-\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(\tilde{x}\right)=0 \tag{21}$$

$$\nabla_{-\mathbf{x}}\times\nabla_{-\mathbf{x}}\times\mathbf{A}\left(\tilde{x}\right)+\frac{\partial}{\partial t}\left(\nabla_{-\mathbf{x}}A^{0}\left(\tilde{x}\right)\right)+\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(\tilde{x}\right)=0 \tag{22}$$

where in the last row we multiplied through by -1, which gives us 19 again in transformed coordinates. Using 3, we get

$$\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\mathbf{A}\left(\tilde{x}\right)-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}A^{0}\left(\tilde{x}\right)\right)+\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(\tilde{x}\right)=0 \tag{23}$$

$$\nabla_{-\mathbf{x}}\times\nabla_{-\mathbf{x}}\times\mathbf{A}\left(\tilde{x}\right)+\frac{\partial}{\partial t}\left(\nabla_{-\mathbf{x}}A^{0}\left(\tilde{x}\right)\right)+\frac{\partial^{2}}{\partial t^{2}}\mathbf{A}\left(\tilde{x}\right)=0 \tag{24}$$

which gives us 19 again in transformed coordinates.

Equation 8 is always true since $\nabla\cdot(\nabla\times\mathbf{A})=0$ for any vector field $\mathbf{A}$, transformed or not.

The last equation 9 is

$$-\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}A^{0}\left(x\right)-\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\times\mathbf{A}\left(x\right)\right)+\frac{\partial}{\partial t}\left(\nabla_{\mathbf{x}}\times\mathbf{A}\left(x\right)\right)=0 \tag{25}$$

$$-\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}A^{0}\left(x\right)=0 \tag{26}$$

Using either 1 or 3 merely affects the sign of A^0, so the last line is valid in either case. Thus all Maxwell's equations for free fields are invariant under parity.

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