PARITY TRANSFORMATION FOR MASSLESS FERMIONS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.6.

By requiring the Dirac Lagrangian to be invariant under parity, L&P show that the parity matrix must satisfy the three conditions

$$\mathsf{P}^{\dagger}\mathsf{P} = 1 \tag{1}$$

$$\gamma_0\mathsf{P}^{\dagger}\gamma_0\gamma_i\mathsf{P} = -\gamma_i \tag{2}$$

$$\gamma_0\mathsf{P}^{\dagger}\gamma_0\mathsf{P} = 1 \tag{3}$$

Since the last condition arises from the mass term in the Lagrangian, if the fermion is massless so that $m = 0$, we need not satisfy this last condition on P. In this case, we can choose

$$\mathsf{P} = \eta_P'\gamma_0\gamma_5 \tag{4}$$

where η_P' is a parity phase factor. We can check that this satisfies the first two conditions by using the properties of the gamma matrices. For the first condition, we have (we'll take $\eta_P' = \pm 1$ in what follows):

$$\mathsf{P}^{\dagger}\mathsf{P} = \gamma_5^{\dagger}\gamma_0^{\dagger}\gamma_0\gamma_5 \tag{5}$$

$$= \gamma_5\gamma_0^2\gamma_5 \tag{6}$$

$$= \gamma_5^2 \tag{7}$$

$$= 1 \tag{8}$$

where we've used the facts that $\gamma_5^{\dagger} = \gamma_5$, $\gamma_0^{\dagger} = \gamma_0$ and $\gamma_5^2 = \gamma_0^2 = 1$.

For the second condition, we have

$$\gamma_0 \mathsf{P}^\dagger \gamma_0 \gamma_i \mathsf{P} = \gamma_0 \gamma_5^\dagger \gamma_0^\dagger \gamma_0 \gamma_i \gamma_0 \gamma_5 \tag{9}$$
$$= \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \tag{10}$$
$$= \gamma_0 \gamma_5 \gamma_0 \gamma_5 \gamma_i \tag{11}$$
$$= -\gamma_0 \gamma_0 \gamma_5 \gamma_5 \gamma_i \tag{12}$$
$$= -\gamma_i \tag{13}$$

In the third line, we used the fact that the anticommutators are$\{\gamma_5, \gamma_i\} = \{\gamma_0, \gamma_i\} = 0$.

We can now see what effect this has on a couple of interaction terms that could appear in the Lagrangian. First, suppose we have the interaction term

$$\mathscr{L}_{\text{int}} = -h\overline{\psi}\psi\phi \tag{14}$$

where ψ is a fermion field and ϕ is a scalar field. The fermion field transforms according to

$$\psi_P(x) = \mathsf{P}\psi(\tilde{x}) \tag{15}$$

where

$$\tilde{x} \equiv (t, -\mathbf{x}) \tag{16}$$

Using 4 we have

$$\mathscr{L}_{\text{int}} \to -h\overline{\psi}_P \psi_P \phi_P \tag{17}$$

For the first factor, we have

$$\overline{\psi}_P(x) = \psi_P^\dagger(\tilde{x})\gamma_0 \tag{18}$$
$$= \psi^\dagger(\tilde{x}) P^\dagger \gamma_0 \tag{19}$$
$$= \overline{\psi}(\tilde{x})\gamma_0 \mathsf{P}^\dagger \gamma_0 \tag{20}$$
$$= \eta_P' \overline{\psi}(\tilde{x})\gamma_0 \gamma_5^\dagger \gamma_0^\dagger \gamma_0 \tag{21}$$
$$= \eta_P' \overline{\psi}(\tilde{x})\gamma_0 \gamma_5^\dagger \tag{22}$$

Therefore

$$\overline{\psi}\psi\phi \to \eta_P'^2 \overline{\psi}(\tilde{x})\gamma_0 \gamma_5^\dagger \gamma_0 \gamma_5 \psi(\tilde{x})\phi(\tilde{x}) \tag{23}$$
$$= -\overline{\psi}(\tilde{x})\gamma_0 \gamma_0 \gamma_5 \gamma_5 \psi(\tilde{x})\phi(\tilde{x}) \tag{24}$$
$$= -\overline{\psi}(\tilde{x})\psi(\tilde{x})\phi(\tilde{x}) \tag{25}$$

Thus for the transformation to be invariant under parity, we must have

$$\phi_P(x) = -\phi(\tilde{x}) \tag{26}$$

which makes ϕ a pseudoscalar field.

For the interaction term

$$\mathscr{L}_{\text{int}} = -h'\overline{\psi}\gamma_5\psi\phi \tag{27}$$

we have

$$\overline{\psi}\gamma_5\psi\phi \to \overline{\psi}(\tilde{x})\gamma_0\gamma_5^\dagger\gamma_5\gamma_0\gamma_5\psi(\tilde{x}) \tag{28}$$

$$= \overline{\psi}(\tilde{x})\gamma_5\psi(\tilde{x}) \tag{29}$$

Thus in this case, for parity invariance we must have

$$\phi_P(x) = +\phi(\tilde{x}) \tag{30}$$

making ϕ a scalar field.

An interaction which combines both of the above interactions, such as

$$\mathscr{L}_{\text{int}} = -\overline{\psi}\left(h + h'\gamma_5\right)\psi\phi \tag{31}$$

can not be parity invariant, as the two terms require opposite parities for ϕ.