CHARGE CONJUGATION AND CURRENTS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 18 Dec 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.9.

W. Greiner & J Reinhardt, *Field Quantization*, (Springer-Verlag, 1996), Chapter 10.

The charge conjugation operation changes the sign of the electric charge of a particle (or, more generally, it changes the signs of electric charge and other quantum numbers of a particle, but we won't go into that here).

L&P show in their section 10.3 that the effect of charge conjugation on a complex scalar field is to interchange the field ϕ with its conjugate $\phi^{\dagger}$. For a real scalar field, $\phi=\phi^{\dagger}$ so a real scalar field is automatically invariant under charge conjugation. For the photon field A^{μ}, the effect of charge conjugation is to map the field into its negative:

$$A_{C}^{\mu}(x)\equiv\mathcal{C}A^{\mu}(x)\mathcal{C}^{-1}=-A^{\mu}(x) \tag{1}$$

This occurs because the interaction term in the electromagnetic Lagrangian is $A_{\mu}j^{\mu}$ where j^{μ} is the 4-current, and reversing the sign of the charges also reverses the current so that $j^{\mu}\to-j^{\mu}$. Thus we must have 1 for charge conjugation invariance.

For the fermion field ψ, it is conventional to write the effect of charge conjugation in the form

$$\psi_{C}\equiv\mathcal{C}\psi(x)\mathcal{C}^{-1}=\mathsf{C}\gamma_{0}^{\mathsf{T}}\psi^{*} \tag{2}$$

where C is a 4×4 matrix and

$$\psi^{*}=\left(\psi^{\dagger}\right)^{\mathsf{T}} \tag{3}$$

where the superscript T represents the matrix transpose.

L&P show that ψ_{C} behaves correctly under a Lorentz transformation if we choose C so that it is a unitary matrix that satisfies

$$\mathsf{C}^{-1}\gamma_{\mu}\mathsf{C}=-\gamma_{\mu}^{\mathsf{T}} \tag{4}$$

In exercise 10.9, L&P ask us to show that the fermion current $\overline{\psi}\gamma_\mu\psi$ becomes its negative under charge conjugation. However, according to Greiner & Reinhardt's equation 10.88, this is actually not true, as we can see by calculating it, as follows.

$$\mathcal{C}\overline{\psi}\gamma_\mu\psi\mathcal{C}^{-1}=\overline{\psi}_C\gamma_\mu\psi_C \tag{5}$$

We can get $\overline{\psi}_C$ by using 4.

$$\begin{aligned}
\overline{\psi}_C &= \psi_C^\dagger\gamma_0 && (6)\\
&= \psi^\mathsf{T}\gamma_0^\mathsf{T}\mathsf{C}^\dagger\gamma_0 && (7)\\
&= \psi^\mathsf{T}\gamma_0^\mathsf{T}\mathsf{C}^{-1}\gamma_0\mathsf{C}\mathsf{C}^{-1} && (8)\\
&= -\psi^\mathsf{T}\gamma_0^\mathsf{T}\gamma_0^\mathsf{T}\mathsf{C}^{-1} && (9)\\
&= -\psi^\mathsf{T}\mathsf{C}^{-1} && (10)
\end{aligned}$$

In the third line, we used the fact that C is unitary, so $\mathsf{C}^\dagger=\mathsf{C}^{-1}$, and to get the last line, we used $\gamma_0^2=1$. Applying this and 2 to 5 we have

$$\begin{aligned}
\mathcal{C}\overline{\psi}\gamma_\mu\psi\mathcal{C}^{-1} &= -\psi^\mathsf{T}\mathsf{C}^{-1}\gamma_\mu\mathsf{C}\gamma_0^\mathsf{T}\psi^* && (11)\\
&= \psi^\mathsf{T}\gamma_\mu^\mathsf{T}\gamma_0^\mathsf{T}\left(\psi^\dagger\right)^\mathsf{T} && (12)\\
&= \left(\psi^\dagger\gamma_0\gamma_\mu\psi\right)^\mathsf{T} && (13)\\
&= \left(\overline{\psi}\gamma_\mu\psi\right)^\mathsf{T} && (14)
\end{aligned}$$

Since each component μ of the original current $\overline{\psi}\gamma_\mu\psi$ is the product of a 1×4 matrix, a 4×4 matrix and a 4×1 matrix, it is just a scalar, so is equal to its transpose. In other words, charge conjugation leaves the fermion current unchanged.

Greiner & Reinhardt show that we can get the correct behaviour for the current if we antisymmetrize it, which means we define it as

$$\begin{aligned}
j^\mu &= \frac{1}{2}\left[\overline{\psi},\gamma^\mu\psi\right] && (15)\\
&= \frac{1}{2}\left(\overline{\psi}\gamma^\mu\psi-\gamma^\mu\psi\overline{\psi}\right) && (16)
\end{aligned}$$

Note that in order for this commutator to make sense, we must interpret it for each matrix component, rather than as a matrix equation, in the same way that we did when calculating fermion anticommutators earlier. That is, we need to write it out using components. If we represent the matrix components by subscripts a and b, with implied sums over repeated indices, we have

$$\left[\overline{\psi},\gamma^{\mu}\psi\right]=\left[\overline{\psi}_{a},\gamma_{ab}^{\mu}\psi_{b}\right] \tag{17}$$

$$=\overline{\psi}_{a}\gamma_{ab}^{\mu}\psi_{b}-\gamma_{ab}^{\mu}\psi_{b}\overline{\psi}_{a} \tag{18}$$

We can now follow the derivation in Greiner & Reinhardt's equation 10.87, using 2, 4 and 10:

$$\mathcal{C}\left[\overline{\psi}_{a},\gamma_{ab}^{\mu}\psi_{b}\right]\mathcal{C}^{-1}=\left[-\psi_{c}^{\mathsf{T}}\mathsf{C}_{ca}^{-1},\gamma_{ad}^{\mu}\mathsf{C}_{de}\left(\gamma_{0}^{\mathsf{T}}\right)_{eb}\psi_{b}^{*}\right] \tag{19}$$

$$=\mathsf{C}_{ca}^{-1}\gamma_{ad}^{\mu}\mathsf{C}_{de}\left[-\psi_{c}^{\mathsf{T}},\left(\gamma_{0}^{\mathsf{T}}\right)_{eb}\psi_{b}^{*}\right] \tag{20}$$

$$=\mathsf{C}_{ca}^{-1}\gamma_{ad}^{\mu}\mathsf{C}_{de}\left[-\psi_{c},\left(\gamma_{0}^{\mathsf{T}}\right)_{eb}\psi_{b}^{\dagger}\right] \tag{21}$$

$$=-\left(\gamma^{\mu}\right)_{ce}^{\mathsf{T}}\left[-\psi_{c},\left(\gamma_{0}^{\mathsf{T}}\right)_{eb}\psi_{b}^{\dagger}\right] \tag{22}$$

$$=-\gamma_{ec}^{\mu}\left[-\psi_{c},\psi_{b}^{\dagger}\left(\gamma_{0}\right)_{be}\right] \tag{23}$$

$$=\gamma_{bc}^{\mu}\left[\psi_{c},\overline{\psi}_{b}\right] \tag{24}$$

$$=\left[\gamma_{bc}^{\mu}\psi_{c},\overline{\psi}_{b}\right] \tag{25}$$

$$=-\left[\overline{\psi}_{b},\gamma_{bc}^{\mu}\psi_{c}\right] \tag{26}$$

Since we're dealing with individual elements of matrices in this derivation, the order of multiplication doesn't matter (except for the field operators ψ, which don't commute). We've also used the fact that for any matrix A, $\mathsf{A}_{ij}^{\mathsf{T}}=\mathsf{A}_{ji}$, and for vectors, $\psi_{i}^{\mathsf{T}}=\psi_{i}$, that is, the ith component is the same regardless of whether we view the vector as a row vector of a column vector. Thus for an antisymmetrized current, charge conjugation does indeed change the sign of the current. I'm not entirely clear why we need to do it this way, although Greiner & Reinhardt say that the difference between the two results 14 and 26 has to do with the anticommutator of ψ and $\overline{\psi}$.

Finally, we can verify that charge conjugation does indeed change the sign of the scalar current, which is

$$j^{\mu}=iq\left(\phi^{\dagger}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{\dagger}\right) \tag{27}$$

Charge conjugation amounts to swapping ϕ and $\phi^{\dagger}$, which obviously converts j^{μ} to $-j^{\mu}$.

PINGBACKS

Pingback: Charge conjugation matrix: some properties
Pingback: Charge conjugation matrix: explicit representations
Pingback: Charge conjugation of Dirac spinors
Pingback: Charge conjugation in quantum electrodynamics
Pingback: Time reversal in the Dirac equation
Pingback: Time reversal in fermion interactions
Pingback: CP transformations in fermion interactions
Pingback: CP transformations of interaction matrices