CHARGE CONJUGATION MATRIX: SOME PROPERTIES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercises 10.10 - 10.12.

For fermion fields, we've seen that charge conjugation can be implemented by using a unitary matrix C that satisfies

$$\mathsf{C}^{-1}\gamma_{\mu}\mathsf{C}=-\gamma_{\mu}^{\mathsf{T}} \tag{1}$$

where γ_{μ} are the Dirac gamma matrices and the superscript T indicates the matrix transpose. Here we'll look at a few properties of the matrix C.

First, we'll look at $\mathsf{C}^{-1}\gamma_5\mathsf{C}$. From the definition of γ_5 we have

$$\begin{aligned}
\mathsf{C}^{-1}\gamma_5\mathsf{C} &= i\mathsf{C}^{-1}\gamma_0\gamma_1\gamma_2\gamma_3\mathsf{C} &(2)\\
&= i\mathsf{C}^{-1}\gamma_0\mathsf{C}\mathsf{C}^{-1}\gamma_1\mathsf{C}\mathsf{C}^{-1}\gamma_2\mathsf{C}\mathsf{C}^{-1}\gamma_3\mathsf{C} &(3)\\
&= (-1)^4\, i\gamma_0^{\mathsf{T}}\gamma_1^{\mathsf{T}}\gamma_2^{\mathsf{T}}\gamma_3^{\mathsf{T}} &(4)\\
&= i\left(\gamma_3\gamma_2\gamma_1\gamma_0\right)^{\mathsf{T}} &(5)\\
&= -i\left(\gamma_0\gamma_3\gamma_2\gamma_1\right)^{\mathsf{T}} &(6)\\
&= -i\left(\gamma_0\gamma_1\gamma_3\gamma_2\right)^{\mathsf{T}} &(7)\\
&= i\left(\gamma_0\gamma_1\gamma_2\gamma_3\right)^{\mathsf{T}} &(8)\\
&= \gamma_5^{\mathsf{T}} &(9)
\end{aligned}$$

where we've the fact that different gamma matrices anticommute.

Next, we can show that C must be antisymmetric. Since charge conjugation applied twice must give back the original field, we have

$$(\psi_C)_C=\psi \tag{10}$$

where ψ_C is the charge conjugated field obtained from the original field ψ. For the fermion field, we have

$$\psi_C=\mathsf{C}\gamma_0^{\mathsf{T}}\psi^* \tag{11}$$

where

$$\psi^* = \left(\psi^\dagger\right)^{\mathsf{T}} \tag{12}$$

We therefore have

$$\begin{aligned}
(\psi_C)_C &= \left(\mathsf{C}\gamma_0^{\mathsf{T}}\left(\psi^\dagger\right)^{\mathsf{T}}\right)_C & (13)\\
&= \mathsf{C}\left(\psi^\dagger\gamma_0\right)_C^{\mathsf{T}} & (14)\\
&= \mathsf{C}\overline{\psi}_C^{\mathsf{T}} & (15)
\end{aligned}$$

We can now use the relation

$$\overline{\psi}_C = -\psi^{\mathsf{T}}\mathsf{C}^{-1} \tag{16}$$

so we have

$$\begin{aligned}
(\psi_C)_C &= -\mathsf{C}\left(\psi^{\mathsf{T}}\mathsf{C}^{-1}\right)^{\mathsf{T}} & (17)\\
&= -\mathsf{C}\left(\mathsf{C}^{-1}\right)^{\mathsf{T}}\psi & (18)
\end{aligned}$$

Since this must be equal to ψ, we must have

$$-\mathsf{C}\left(\mathsf{C}^{-1}\right)^{\mathsf{T}} = I \tag{19}$$

or, since $\left(\mathsf{C}^{-1}\mathsf{C}\right)^{\mathsf{T}} = \mathsf{C}^{\mathsf{T}}\left(\mathsf{C}^{-1}\right)^{\mathsf{T}} = I$, we must have

$$\mathsf{C} = -\mathsf{C}^{\mathsf{T}} \tag{20}$$

which is the condition for C to be antisymmetric.

Finally, suppose we have two different representations of the gamma matrices that are related by the unitary transformaton

$$\tilde{\gamma}_\mu = U\gamma_\mu U^\dagger \tag{21}$$

What are the charge conjugation matrices in the new representation? We require 1 to be true in the new representation, so, using $U^\dagger = U^{-1}$ for a unitary matrix,

$$\tilde{\mathsf{C}}^{-1}\tilde{\gamma}_{\mu}\tilde{\mathsf{C}} = -\tilde{\gamma}_{\mu}^{\mathsf{T}} \tag{22}$$

$$= -\left(U^{\dagger}\right)^{\mathsf{T}}\gamma_{\mu}^{\mathsf{T}}U^{\mathsf{T}} \tag{23}$$

$$= \left(U^{\dagger}\right)^{\mathsf{T}}\mathsf{C}^{-1}\gamma_{\mu}\mathsf{C}U^{\mathsf{T}} \tag{24}$$

$$= \left(U^{\dagger}\right)^{\mathsf{T}}\mathsf{C}^{-1}U^{\dagger}U\gamma_{\mu}U^{\dagger}U\mathsf{C}U^{\mathsf{T}} \tag{25}$$

$$= \left(U^{\dagger}\right)^{\mathsf{T}}\mathsf{C}^{-1}U^{\dagger}\tilde{\gamma}_{\mu}U\mathsf{C}U^{\mathsf{T}} \tag{26}$$

By comparing this with the original LHS ($\tilde{\mathsf{C}}^{-1}\tilde{\gamma}_{\mu}\tilde{\mathsf{C}}$) we see that the relation is satisfied if

$$\tilde{\mathsf{C}} = U\mathsf{C}U^{\mathsf{T}} \tag{27}$$

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