CHARGE CONJUGATION MATRIX: EXPLICIT REPRESENTATIONS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.13.

For fermion fields, we've seen that charge conjugation can be implemented by using a unitary matrix C that satisfies

$$\mathsf{C}^{-1}\gamma^{\mu}\mathsf{C}=-\gamma^{\mu\mathsf{T}} \tag{1}$$

where γ_{μ} are the Dirac gamma matrices and the superscript T indicates the matrix transpose. We can find explicit representations of C for particular representations of the gamma matrices.

One representation is the Dirac-Pauli representation, which is

$$\gamma^{0}=\left[\begin{array}{cc} I & 0\\ 0 & -I \end{array}\right] \tag{2}$$

$$\gamma^{i}=\left[\begin{array}{cc} 0 & \sigma^{i}\\ -\sigma^{i} & 0 \end{array}\right] \tag{3}$$

where the σ^{i} are the Pauli matrices

$$\sigma^{1}=\left[\begin{array}{cc} 0 & 1\\ 1 & 0 \end{array}\right];\quad\sigma^{2}=\left[\begin{array}{cc} 0 & -i\\ i & 0 \end{array}\right];\quad\sigma^{3}=\left[\begin{array}{cc} 1 & 0\\ 0 & -1 \end{array}\right] \tag{4}$$

We can verify by direct calculation that if we take

$$\mathsf{C}=i\gamma^{2}\gamma^{0} \tag{5}$$

then 1 is satisfied. We have

$$\mathsf{C} = i\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{6}$$

$$= i\begin{bmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

We have (multiply $\mathsf{C}\mathsf{C}^{-1}$ out to see this)

$$\mathsf{C}^{-1} = -\mathsf{C} = -i\gamma^2\gamma^0 \tag{9}$$

We can now check 1.

$$\mathsf{C}^{-1}\gamma^\mu\mathsf{C} = \gamma^2\gamma^0\gamma^\mu\gamma^2\gamma^0 \tag{10}$$

If $\mu = 1$ or $\mu = 3$, we have, using the anticommutators and squares of the gamma matrices

$$\gamma^2\gamma^0\gamma^\mu\gamma^2\gamma^0 = \gamma^\mu\gamma^2\gamma^0\gamma^2\gamma^0 \tag{11}$$

$$= -\gamma^\mu\gamma^2\gamma^2\gamma^0\gamma^0 \tag{12}$$

$$= \gamma^\mu \tag{13}$$

From 3 and 4 we see that

$$\gamma^1 = -\gamma^{1\mathsf{T}} \tag{14}$$

$$\gamma^3 = -\gamma^{3\mathsf{T}} \tag{15}$$

so 1 is true for $\mu = 1,3$.

For $\mu = 0$, we have

$$\mathsf{C}^{-1}\gamma^0\mathsf{C} = \gamma^2\gamma^0\gamma^0\gamma^2\gamma^0 \tag{16}$$

$$= \gamma^2\gamma^2\gamma^0 \tag{17}$$

$$= -\gamma^0 \tag{18}$$

$$= -\gamma^{0\mathsf{T}} \tag{19}$$

where the last line follows because γ^0 is symmetric, so $\gamma^{0\mathsf{T}} = \gamma^0$.

For $\mu = 2$, we have, since γ^2 is also symmetric, as can be seen from 3 and 4:

$$\mathsf{C}^{-1}\gamma^2\mathsf{C} = \gamma^2\gamma^0\gamma^2\gamma^2\gamma^0 \tag{20}$$
$$= -\gamma^2\gamma^0\gamma^0 \tag{21}$$
$$= -\gamma^2 \tag{22}$$
$$= -\gamma^{2\mathsf{T}} \tag{23}$$

Thus 1 is valid for all γ^μ in the Dirac-Pauli representation.

Another representation is the Majorana representation, which is given in L&P's Appendix A, equation A.11:

$$\gamma^0 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix} \tag{24}$$
$$\gamma^1 = \begin{bmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{bmatrix} \tag{25}$$
$$\gamma^2 = \begin{bmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{bmatrix} \tag{26}$$
$$\gamma^3 = \begin{bmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{bmatrix} \tag{27}$$

In the Majorana representation, we have

$$\mathsf{C} = i\gamma^0 \tag{28}$$
$$\mathsf{C}^{-1} = -i\gamma^0 \tag{29}$$

We get

$$\mathsf{C}^{-1}\gamma^0\mathsf{C} = \gamma^0\gamma^0\gamma^0 \tag{30}$$
$$= \gamma^0 \tag{31}$$
$$= -\gamma^{0\mathsf{T}} \tag{32}$$

by comparison with 24.

For $\mu = 1,2,3 = j$ we have

$$\mathsf{C}^{-1}\gamma^j\mathsf{C} = \gamma^0\gamma^j\gamma^0 \tag{33}$$
$$= -\gamma^j\gamma^0\gamma^0 \tag{34}$$
$$= -\gamma^j \tag{35}$$
$$= -\gamma^{j\mathsf{T}} \tag{36}$$

where the last line follows because all 3 of the γ^j are symmetric.

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