CHARGE CONJUGATION OF DIRAC SPINORS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.14.

As shown in L&P's equation 10.37, charge conjugation of a fermion field ψ is defined to give a new field ψ_C such that

$$\psi_C = \mathsf{C}\gamma_0^{\mathsf{T}}\psi^* \tag{1}$$

where C is the unitary charge conjugation matrix and

$$\psi^* \equiv \psi^{\dagger\mathsf{T}} \tag{2}$$

In the Dirac-Pauli representation, C has the form

$$\mathsf{C} = i\gamma^2\gamma^0 \tag{3}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Using this form, we can find the effect of charge conjugation on the Dirac spinors, which are

$$u_\pm(\mathbf{p}) = \sqrt{E_p+m}\begin{bmatrix} \chi_\pm \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\pm \end{bmatrix} \tag{5}$$

$$v_\pm(\mathbf{p}) = \pm\sqrt{E_p+m}\begin{bmatrix} \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_\mp \\ \chi_\mp \end{bmatrix} \tag{6}$$

where the σ^i are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

and

$$\chi_{+}=\left[\begin{array}{c}1\\0\end{array}\right] \tag{8}$$

$$\chi_{-}=\left[\begin{array}{c}0\\1\end{array}\right] \tag{9}$$

We also have

$$\gamma_{0}^{\mathsf{T}}=\gamma_{0}=\left[\begin{array}{cc}I & 0\\0 & -I\end{array}\right] \tag{10}$$

$$\gamma^{2}=\left[\begin{array}{cc}0 & \sigma^{2}\\-\sigma^{2} & 0\end{array}\right] \tag{11}$$

$$\mathsf{C}=i\left[\begin{array}{cc}0 & -\sigma^{2}\\-\sigma^{2} & 0\end{array}\right] \tag{12}$$

so

$$\mathsf{C}\gamma_{0}^{\mathsf{T}}=i\left[\begin{array}{cc}0 & \sigma^{2}\\-\sigma^{2} & 0\end{array}\right] \tag{13}$$

We can now do some actual calculations. Consider first

$$\mathsf{C}\gamma_{0}^{\mathsf{T}}u_{+}^{*}=i\sqrt{E_{p}+m}\left[\begin{array}{cc}0 & \sigma^{2}\\-\sigma^{2} & 0\end{array}\right]\left[\begin{array}{c}\chi_{+}\\\frac{\sigma\cdot\mathbf{p}}{E_{p}+m}\chi_{+}\end{array}\right]^{*} \tag{14}$$

where the * on the spinor now indicates just complex conjugate (no transpose). The complex conjugate affects only the term involving σ^{2} since all the other terms are real. In this case, from 7 we have

$$\sigma^{2*}=-\sigma^{2} \tag{15}$$

We therefore have

$$\mathsf{C}\gamma_{0}^{\mathsf{T}}u_{+}^{*}=\sqrt{E_{p}+m}\left[\begin{array}{c}i\sigma^{2}\frac{\sigma^{*}\cdot\mathbf{p}}{E_{p}+m}\chi_{+}\\-i\sigma^{2}\chi_{+}\end{array}\right] \tag{16}$$

The bottom entry is

$$-i\sigma^{2}\chi_{+}=\left[\begin{array}{cc}0 & -1\\1 & 0\end{array}\right]\left[\begin{array}{c}1\\0\end{array}\right]=\left[\begin{array}{c}0\\1\end{array}\right]=\chi_{-} \tag{17}$$

For the top entry, we can use the anticommutators of the Pauli matrices, namely for $i\neq j$

$$\sigma^{i}\sigma^{j}=-\sigma^{j}\sigma^{i} \tag{18}$$

If we now look at the top entry in 16 and look specifically at the terms involving the σ^i, we have

$$\sigma^2(\sigma^*\cdot\mathbf{p}) = \sigma^2\left(\sigma^1p^1 - \sigma^2p^2 + \sigma^3p^3\right) \tag{19}$$
$$= -\left(\sigma^1p^1 + \sigma^2p^2 + \sigma^3p^3\right)\sigma^2 \tag{20}$$
$$= -\sigma\cdot\mathbf{p}\sigma^2 \tag{21}$$

Inserting this back into 16 we have

$$i\sigma^2\frac{\sigma^*\cdot\mathbf{p}}{E_p+m}\chi_+ = -\frac{\sigma\cdot\mathbf{p}}{E_p+m}i\sigma^2\chi_+ \tag{22}$$
$$= \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_- \tag{23}$$

Comparing with 6 we see that

$$\mathsf{C}\gamma_0^{\mathsf{T}}u_+^* = \sqrt{E_p+m}\begin{bmatrix}\frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_-\\ \chi_-\end{bmatrix} = v_+ \tag{24}$$

We can do a similar calculation for u_- to find that

$$\mathsf{C}\gamma_0^{\mathsf{T}}u_-^* = v_- \tag{25}$$

To transform $v_\pm$, consider v_+. We have

$$\mathsf{C}\gamma_0^{\mathsf{T}}v_+^* = i\sqrt{E_p+m}\begin{bmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{bmatrix}\begin{bmatrix}\frac{\sigma^*\cdot\mathbf{p}}{E_p+m}\chi_-\\ \chi_-\end{bmatrix} \tag{26}$$
$$= \sqrt{E_p+m}\begin{bmatrix}i\sigma^2\chi_-\\ -i\sigma^2\frac{\sigma^*\cdot\mathbf{p}}{E_p+m}\chi_-\end{bmatrix} \tag{27}$$

We now use

$$i\sigma^2\chi_- = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = \chi_+ \tag{28}$$

We then get

$$\mathsf{C}\gamma_0^{\mathsf{T}}v_+^* = \sqrt{E_p+m}\begin{bmatrix}\chi_+\\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}i\sigma^2\chi_-\end{bmatrix} \tag{29}$$
$$= \sqrt{E_p+m}\begin{bmatrix}\chi_+\\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_+\end{bmatrix} \tag{30}$$
$$= u_+ \tag{31}$$

Another similar calculation gives

$$\mathsf{C}\gamma_0^{\mathsf{T}} v_-^* = u_- \tag{32}$$