TIME REVERSAL IN THE DIRAC EQUATION

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.16.

We've had a look at time reversal earlier, where we saw that it is an antilinear operator. A system is time reversal invariant if it remains the same when we keep the spatial coordinates the same and reverse the direction of time, so that $t \to -t$. The coordinates in the time reversed system are thus

$$x_T \equiv (-t, \mathbf{x}) = -\tilde{x} \tag{1}$$

where $\tilde{x}$ is defined by this equation.

The original Dirac Lagrangian is

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{2}$$

in which we can separate the time and space coordinates to give

$$\mathscr{L} = \psi^{\dagger}(x)\left(i\partial_t + i\gamma_0\gamma\cdot\nabla_x - m\gamma_0\right)\psi(x) \tag{3}$$

In section 10.4, L&P derive the time-reversed Dirac Lagrangian.

$$\mathcal{T}\mathscr{L}(x)\mathcal{T}^{-1} = \psi_T^{\dagger}(x)\left(-i\partial_t - i\gamma_0^*\gamma^*\cdot\nabla_x - m\gamma_0^*\right)\psi_T(x) \tag{4}$$

$$= \psi_T^{\dagger}(x)\left(-i\partial_t + i\gamma_0^{\mathsf{T}}\gamma^{\mathsf{T}}\cdot\nabla_x - m\gamma_0^{\mathsf{T}}\right)\psi_T(x) \tag{5}$$

$$\psi^{\dagger}(-\tilde{x})\,\mathsf{T}^{\dagger}\left(i\partial_{-t} + i\gamma_0^{\mathsf{T}}\gamma^{\mathsf{T}}\cdot\nabla_x - m\gamma_0^{\mathsf{T}}\right)\mathsf{T}\psi(-\tilde{x}) \tag{6}$$

In the first line, we use the antilinearity of time reversal, so that all values are replaced by their complex conjugates. We also note that for a square matrix such as γ_μ, taking the hermitian conjugate followed by the transpose (which is the meaning L&P give to the superscript * when applied to a matrix) is equivalent to just taking the complex conjugate, since the hermitian conjugate is complex conjugate followed by the transpose, so the two transpose operations cancel out. Thus $\gamma_0^* = \gamma_0^{\dagger\mathsf{T}}$ is also just the complex conjugate of γ_0.

To get the second line, we use the hermiticity property of γ^μ (L&P equation 4.18) and anticommutators:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{7}$$

Thus

$$\gamma_0^* = \gamma_0^{\dagger\mathsf{T}} = \gamma_0^{\mathsf{T}} \tag{8}$$
$$\gamma_i^* = \gamma_i^{\dagger\mathsf{T}} = (\gamma_0\gamma_i\gamma_0)^{\mathsf{T}} = -(\gamma_0\gamma_0\gamma_i)^{\mathsf{T}} = -\gamma_i^{\mathsf{T}} \tag{9}$$

In the last line, the 4×4 matrix T is used to generate the time reversed Dirac field according to

$$\psi_T(x) = \mathsf{T}\psi(-\tilde{x}) \tag{10}$$

Time reversal invariance of the Lagrangian requires that 6 be the same as the original Lagrangian with x replaced by $-\tilde{x}$, so that

$$\mathcal{T}\mathscr{L}(x)\mathcal{T}^{-1} = \mathscr{L}(-\tilde{x}) \tag{11}$$

This gives three conditions on the matrix T:

Remember that T as a superscript indicates a matrix transpose, and is not the same as the 4×4 matrix T.

$$\mathsf{T}^\dagger\mathsf{T} = 1 \tag{12}$$
$$\mathsf{T}^\dagger\gamma_0^{\mathsf{T}}\gamma_i^{\mathsf{T}}\mathsf{T} = \gamma_0\gamma_i \tag{13}$$
$$\mathsf{T}^\dagger\gamma_0^{\mathsf{T}}\mathsf{T} = \gamma_0 \tag{14}$$

Using the relation

$$\mathsf{C}^{-1}\gamma_\mu\mathsf{C} = -\gamma_\mu^{\mathsf{T}} \tag{15}$$

we can get rid of the matrix transposes in these conditions. From 14 we have (since T is antiunitary, we have $\mathsf{T}^{-1} = \mathsf{T}^\dagger$):

$$\mathsf{T}^\dagger\gamma_0^{\mathsf{T}}\mathsf{T} = \mathsf{T}^{-1}\gamma_0^{\mathsf{T}}\mathsf{T} \tag{16}$$
$$= -\mathsf{T}^{-1}\mathsf{C}^{-1}\gamma_0\mathsf{C}\mathsf{T} \tag{17}$$
$$= \gamma_0 \tag{18}$$

or

$$\mathsf{C}\mathsf{T}\gamma_0 = -\gamma_0\mathsf{C}\mathsf{T} \tag{19}$$

which gives the anticommutator

$$\{\mathsf{C}\mathsf{T},\gamma_0\} = 0 \tag{20}$$

Now, using 13 we have

$$\mathsf{T}\gamma_0\gamma_i\mathsf{T}^{-1} = \gamma_0^{\mathsf{T}}\gamma_i^{\mathsf{T}} \tag{21}$$
$$= \mathsf{C}^{-1}\gamma_0\mathsf{C}\mathsf{C}^{-1}\gamma_i\mathsf{C} \tag{22}$$
$$= \mathsf{C}^{-1}\gamma_0\gamma_i\mathsf{C} \tag{23}$$

This gives

$$\mathsf{CT}\gamma_0\gamma_i = \gamma_0\gamma_i\mathsf{CT} \tag{24}$$
$$-\gamma_0\mathsf{CT}\gamma_i = \gamma_0\gamma_i\mathsf{CT} \tag{25}$$

We now multiply on the left by γ_0 to get

$$-\mathsf{CT}\gamma_i = \gamma_i\mathsf{CT} \tag{26}$$
$$\{\mathsf{CT},\gamma_i\} = 0 \tag{27}$$

Thus CT anticommutes with all the gamma matrices:

$$\{\mathsf{CT},\gamma_\mu\} = 0 \tag{28}$$

As the matrix γ_5 also anticommutes with all the gamma matrices, this makes CT a multiple of γ_5, which is written as

$$\mathsf{T} = \eta_T\mathsf{C}^{-1}\gamma_5 \tag{29}$$

where η_T is the time reversal phase factor. Referring back to 10, we thus have for a Dirac field:

$$\psi_T(x) = \eta_T\mathsf{C}^{-1}\gamma_5\psi(-\tilde{x}) \tag{30}$$

We can now see what happens if we have two different representations of the γ_μ related by the unitary transformation

$$\tilde{\gamma}_\mu = U\gamma_\mu U^\dagger \tag{31}$$

The time reversed Dirac Lagrangian using the new gamma matrices is obtained from 5, so we have

$$\mathcal{T}\mathscr{L}(x)\mathcal{T}^{-1}=\tilde{\psi}_T^{\dagger}(x)\left(-i\partial_t+i\tilde{\gamma}_0^{\mathsf{T}}\tilde{\gamma}^{\mathsf{T}}\cdot\nabla_x-m\tilde{\gamma}_0^{\mathsf{T}}\right)\tilde{\psi}_T(x) \tag{32}$$

$$=\tilde{\psi}_T^{\dagger}(x)\left(-i\partial_t+i\left(U\gamma_0U^{\dagger}\right)^{\mathsf{T}}\left(U\gamma U^{\dagger}\right)^{\mathsf{T}}\cdot\nabla_x-m\left(U\gamma_0U^{\dagger}\right)^{\mathsf{T}}\right)\tilde{\psi}_T(x) \tag{33}$$

$$=\tilde{\psi}_T^{\dagger}(x)\left(-i\partial_t+iU^{\dagger\mathsf{T}}\gamma_0^{\mathsf{T}}U^{\mathsf{T}}U^{\dagger\mathsf{T}}\gamma^{\mathsf{T}}U^{\mathsf{T}}\cdot\nabla_x-mU^{\dagger\mathsf{T}}\gamma_0^{\mathsf{T}}U^{\mathsf{T}}\right)\tilde{\psi}_T(x) \tag{34}$$

$$=\tilde{\psi}_T^{\dagger}(x)\left(-U^{\dagger\mathsf{T}}U^{\mathsf{T}}i\partial_t+iU^{\dagger\mathsf{T}}\gamma_0^{\mathsf{T}}\gamma^{\mathsf{T}}U^{\mathsf{T}}\cdot\nabla_x-mU^{\dagger\mathsf{T}}\gamma_0^{\mathsf{T}}U^{\mathsf{T}}\right)\tilde{\psi}_T(x) \tag{35}$$

where we've inserted the identity matrix in the form $U^{\dagger\mathsf{T}}U^{\mathsf{T}}$ in the first term in the last line.

This matches the original time-reversed Lagrangian 5 if

$$\psi_T(x)=U^{\mathsf{T}}\tilde{\psi}_T(x) \tag{36}$$

or, multiplying both sides by $U^{\dagger\mathsf{T}}=U^*$

$$\tilde{\psi}_T(x)=U^*\psi_T(x) \tag{37}$$

PINGBACKS

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