TIME REVERSAL IN FERMION INTERACTIONS

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 10, Exercise 10.17.

Time reversal of a Dirac fieldψ can be represented by the time reversal matrix T as

$$\psi_T(x)=\mathcal{T}\psi(x)\mathcal{T}^{-1}=\mathsf{T}\psi(-\tilde{x}) \tag{1}$$

where the time reversed coordinate system $\tilde{x}$ is given by

$$\tilde{x}=(-t,\mathbf{x}) \tag{2}$$

We've also seen that T can be represented as

$$\mathsf{T}=\eta_T\mathsf{C}^{-1}\gamma_5 \tag{3}$$

where η_T is the time reversal phase factor and C is the charge conjugation matrix.

All this was derived for a free Dirac field, without interactions. If we want to investigate the effect of time reversal on interactions involving fermion fields, we need to consider the usual bilinear term of the form

$$h\overline{\psi}_1F\psi_2 \tag{4}$$

where ψ_1 and ψ_2 are two (not necessarily the same) fermion fields, h is a constant and F is a constant 4×4 matrix. We time-reverse this term as follows.

$$\mathcal{T}h\overline{\psi}_1F\psi_2\mathcal{T}^{-1}=h^*\mathcal{T}\left(\psi_1^\dagger\gamma_0F\psi_2\right)\mathcal{T}^{-1} \tag{5}$$

where h^* is the complex conjugate of h, because time reversal is an antilinear operation. To apply time reversal to the terms inside the parentheses, we recall that, for a square matrix, taking the complex conjugate is the same as taking the hermitian conjugate followed by the transpose, which is represented in L&P by a superscript *. Combining this with 1 we get

$$\mathcal{T}h\overline{\psi}_1 F\psi_2\mathcal{T}^{-1} = h^*\psi_1^\dagger\left(-\tilde{x}\right)\mathsf{T}^\dagger\gamma_0^* F^*\mathsf{T}\psi_2\left(-\tilde{x}\right) \tag{6}$$
$$= h^*\overline{\psi}_1\left(-\tilde{x}\right)\gamma_0\gamma_5\mathsf{C}\gamma_0^{\mathsf{T}}F^*\mathsf{C}^{-1}\gamma_5\psi_2\left(-\tilde{x}\right) \tag{7}$$

where in the last line we've used 3 and the fact that γ_0 is hermitian so that $\gamma_0^* = \gamma_0^{\dagger\mathsf{T}} = \gamma_0^{\mathsf{T}}$. Using the definition of C as

$$\mathsf{C}^{-1}\gamma_\mu\mathsf{C} = -\gamma_\mu^{\mathsf{T}} \tag{8}$$

we have

$$\mathsf{C}\gamma_0^{\mathsf{T}} = -\gamma_0\mathsf{C} \tag{9}$$

and we get final form

$$\mathcal{T}h\overline{\psi}_1 F\psi_2\mathcal{T}^{-1} = h^*\overline{\psi}_1\left(-\tilde{x}\right)F_T\psi_2\left(-\tilde{x}\right) \tag{10}$$

where

$$F_T \equiv \gamma_5\mathsf{C}F^*\mathsf{C}^{-1}\gamma_5 \tag{11}$$

Using 8 and the anticommutators of the gamma matrices, we can work out F_T for the cases where F is some combination of the γ_μ. L&P give the result in equation 10.83, but it's worth working through a couple of examples to see how they get this table.

For $F = 1$, the result is pretty obvious:

$$F_T = \gamma_5\mathsf{C}\mathsf{C}^{-1}\gamma_5 = \gamma_5\gamma_5 = 1 \tag{12}$$

For $F = \gamma_5$ we have, using $\gamma_5^\dagger = \gamma_5$ and the result $\mathsf{C}^{-1}\gamma_5\mathsf{C} = \gamma_5^{\mathsf{T}}$:

$$F_T = \gamma_5\mathsf{C}\gamma_5^{\dagger\mathsf{T}}\mathsf{C}^{-1}\gamma_5 \tag{13}$$
$$= \gamma_5\mathsf{C}\gamma_5^{\mathsf{T}}\mathsf{C}^{-1}\gamma_5 \tag{14}$$
$$= \gamma_5\gamma_5\gamma_5 \tag{15}$$
$$= \gamma_5 \tag{16}$$

One more example. For $F = \gamma_i$ we have, using $\gamma_i^\dagger = \gamma_0\gamma_i\gamma_0$:

$$F_T = \gamma_5 \mathsf{C} \gamma_i^{\dagger\mathsf{T}} \mathsf{C}^{-1} \gamma_5 \tag{17}$$
$$= \gamma_5 \mathsf{C} \gamma_0^{\mathsf{T}} \gamma_i^{\mathsf{T}} \gamma_0^{\mathsf{T}} \mathsf{C}^{-1} \gamma_5 \tag{18}$$
$$= -\gamma_5 \mathsf{C} \gamma_0^{\mathsf{T}} \gamma_0^{\mathsf{T}} \gamma_i^{\mathsf{T}} \mathsf{C}^{-1} \gamma_5 \tag{19}$$
$$= -\gamma_5 \mathsf{C} \gamma_i^{\mathsf{T}} \mathsf{C}^{-1} \gamma_5 \tag{20}$$
$$= +\gamma_5 \gamma_i \gamma_5 \tag{21}$$
$$= -\gamma_i \gamma_5 \gamma_5 \tag{22}$$
$$= -\gamma_i \tag{23}$$

The other entries are found in similar ways, involving juggling gamma matrices. The result is L&P's equation 10.83, which is

F	1	γ_5	γ_0	γ_i	$\gamma_0\gamma_5$	$\gamma_i\gamma_5$
F_T	1	γ_5	γ_0	$-\gamma_i$	$\gamma_0\gamma_5$	$-\gamma_i\gamma_5$

As an example, we consider the interaction term we met earlier

$$\mathscr{L}_{\text{int}} = \overline{\psi}\gamma^\mu \left(a + b\gamma_5\right)\psi Z_\mu \tag{24}$$

where Z_μ is some spin-1 field and a and b are constants. To make this time-reversal invariant, we first need to ensure that the Lagrangian is hermitian. We have, assuming Z_μ is real:

$$\mathscr{L}_{\text{int}}^\dagger = a^*\psi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 Z_\mu + b^*\psi^\dagger\gamma_5\gamma^0\gamma^\mu\gamma^0\gamma^0\psi Z_\mu \tag{25}$$
$$= a^*\psi^\dagger\gamma^0\gamma^\mu Z_\mu + b^*\psi^\dagger\gamma_5\gamma^0\gamma^\mu\psi Z_\mu \tag{26}$$
$$= a^*\psi^\dagger\gamma^0\gamma^\mu Z_\mu + (-1)^2 b^*\psi^\dagger\gamma^0\gamma^\mu\gamma_5\psi Z_\mu \tag{27}$$
$$= \overline{\psi}\gamma^\mu \left(a^* + b^*\gamma_5\right)\psi Z_\mu \tag{28}$$

where in the third line, we anticommuted γ_5 with $\gamma^0\gamma^\mu$, which introduced the factor of $(-1)^2$.

The result is equal to $\mathscr{L}_{\text{int}}$ only if $a^* = a$ and $b^* = b$, so that these two constants must both be real. This means that the antilinearity of time reversal won't affect them. We can now apply the above table to see how $\mathscr{L}_{\text{int}}^\dagger$ transforms under time reversal. We have for the first term in 24

$$\mathcal{T}\left(a\overline{\psi}\gamma^\mu\psi Z_\mu\right)\mathcal{T}^{-1} = a\overline{\psi}\left(-\tilde{x}\right)F_T^\mu\psi\left(-\tilde{x}\right)Z_{\mu T} \tag{29}$$
$$= a\overline{\psi}\left(-\tilde{x}\right)F_T^0\psi\left(-\tilde{x}\right)Z_{0T} + a\overline{\psi}\left(-\tilde{x}\right)F_T^i\psi\left(-\tilde{x}\right)Z_{iT} \tag{30}$$

where F^μ in this case is $F^\mu = \gamma^\mu$. Thus we have

$$\mathcal{T}\left(a\overline{\psi}\gamma^{\mu}\psi Z_{\mu}\right)\mathcal{T}^{-1}=a\overline{\psi}\left(-\tilde{x}\right)\gamma^{0}\psi\left(-\tilde{x}\right)Z_{0T}-a\overline{\psi}\left(-\tilde{x}\right)\gamma^{i}\psi\left(-\tilde{x}\right)Z_{iT} \tag{31}$$

For the second term in 24 we have

$$\mathcal{T}\left(b\overline{\psi}\gamma^{\mu}\gamma_{5}\psi Z_{\mu}\right)\mathcal{T}^{-1}=b\overline{\psi}\left(-\tilde{x}\right)F_{T}^{0}\psi\left(-\tilde{x}\right)Z_{0T}+b\overline{\psi}\left(-\tilde{x}\right)F_{T}^{i}\psi\left(-\tilde{x}\right)Z_{iT} \tag{32}$$

where F^{μ} in this case is $F^{\mu}=\gamma^{\mu}\gamma_{5}$. Thus we have

$$\mathcal{T}\left(b\overline{\psi}\gamma^{\mu}\gamma_{5}\psi Z_{\mu}\right)\mathcal{T}^{-1}=b\overline{\psi}\left(-\tilde{x}\right)\gamma^{0}\gamma_{5}\psi\left(-\tilde{x}\right)Z_{0T}-b\overline{\psi}\left(-\tilde{x}\right)\gamma^{i}\gamma_{5}\psi\left(-\tilde{x}\right)Z_{iT} \tag{33}$$

In both terms, we see that we can obtain time reversal invariance if

$$Z_{0T}=Z_{0} \tag{34}$$

$$Z_{iT}=-Z_{i} \tag{35}$$

which is the same time-reversal behaviour as the photon field A_{μ}.