GENERAL ELECTROMAGNETIC VERTEX

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 11, Exercise 11.1.

I should begin this post with a warning that I'm not entirely certain of some of the steps and concepts in L&P's section 11.1, but I'll explain things as far as I've managed to get.

They begin by writing the matrix element for the free-field electromagnetic current j_μ between two fermion states $|\mathbf{p},s\rangle$ and $|\mathbf{p}',s'\rangle$. Using

$$j_\mu(x) = eQ\overline{\psi}\gamma_\mu\psi \tag{1}$$

the matrix element is

$$\left\langle \mathbf{p}',s'\left|j_\mu(x)\right|\mathbf{p},s\right\rangle = \frac{e^{-iq\cdot x}}{\sqrt{2E_pV}\sqrt{2E_{p'}V}}\overline{u}_{s'}\left(\mathbf{p}'\right)eQ\gamma_\mu u_s(\mathbf{p}) \tag{2}$$

This form uses the results obtained earlier

$$\psi_+(x)\left|e^-(\mathbf{k},s)\right\rangle = \frac{1}{\sqrt{2E_kV}}e^{-ik\cdot x}u_s(\mathbf{k})|0\rangle \tag{3}$$

$$\left\langle e^-(\mathbf{k},s)\right|\overline{\psi}_-(x) = \frac{1}{\sqrt{2E_kV}}e^{ik\cdot x}\overline{u}_s(\mathbf{k})\langle 0| \tag{4}$$

When the current interacts with other fields, 1 is no longer a valid way of describing the current, since it relies on the free field ψ. The actual form of $j_\mu(x)$ in this case isn't specified, but presumably it can involve fields other than the Dirac fields.

To proceed, L&P say that we can use the fact that the momentum operator $\mathscr{P}_\mu$ generates space-time translations, so we have

$$j_\mu(x) = e^{\mathscr{P}\cdot x}j_\mu(0)\,e^{-i\mathscr{P}\cdot x} \tag{5}$$

Inserting this into the LHS of 2 we get

$$\left\langle \mathbf{p}',s'\left|j_\mu(x)\right|\mathbf{p},s\right\rangle = e^{-iq\cdot x}\left\langle \mathbf{p}',s'\left|j_\mu(0)\right|\mathbf{p},s\right\rangle \tag{6}$$

where

$$q \equiv p - p' \tag{7}$$

since the momentum operators in 5 act on the bra and ket to give the exponential $e^{-iq\cdot x}$.

What happens next is a bit of a mystery. L&P say that we can parametrize $\langle \mathbf{p}', s' | j_\mu(0) | \mathbf{p}, s\rangle$, presumably in analogy with 2, so that instead of the $Q\gamma_\mu$in 2, we have a term $\Gamma_\mu(p, p')$:

$$\left\langle \mathbf{p}', s' | j_\mu(x) | \mathbf{p}, s \right\rangle = \frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\sqrt{2E_{p'}V}} \overline{u}_{s'}(\mathbf{p}') \, e\Gamma_\mu(p, p') \, u_s(\mathbf{p}) \tag{8}$$

As far as I can tell, this is just a postulate, and can't actually be derived from anything else.

From the form of 8 and the fact that the LHS is just a number for each component μ, we see that Γ_μ must be a 4×4 matrix, since $\overline{u}_{s'}$ is a 4-component row vector and u_s is a 4-component column vector. This also follows from the analogy with 2, where γ_μ is a 4×4 matrix.

To get constraints on possible values for Γ_μ, we note first that it must have a single free vector index μ. L&P then state that the most general form for Γ_μ is their equation 11.6:

$$\begin{aligned}\Gamma_\mu = {} & \gamma_\mu\left(F_1 + \tilde{F}_1\gamma_5\right) + \left(iF_2 + \tilde{F}_2\gamma_5\right)\sigma_{\mu\nu}q^\nu \\ & \tilde{F}_3 q_\mu \not{q}\gamma_5 + q_\mu\left(F_4 + \tilde{F}_4\gamma_5\right)\end{aligned} \tag{9}$$

One's first reaction is probably, 'where did that come from?' To understand it, we need to return to a result derived earlier, where we showed that the set of matrices

$$\left\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5\right\} \tag{10}$$

where

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = -\sigma^{\nu\mu} \tag{11}$$

forms a basis for all 4×4 matrices. Therefore, the most general form for Γ_μ is a linear combination of these 5 matrices. We need to combine this fact with the requirement that Γ_μ must have a single vector index μ and that it depends on p_μ and p'_μ. We can express the dependence on p_μ and p'_μ as dependence on $(p+p')_\mu$ and $(p-p')_\mu = q_\mu$. We can then use the Gordon identity

$$\overline{u}'\gamma^\mu u = \frac{1}{2m}\overline{u}'\left[\left(p+p'\right)^\mu - i\sigma^{\mu\alpha}q_\alpha\right]u \tag{12}$$

to eliminate $(p+p')_\mu$ in favour of γ_μ and $\sigma_{\mu\nu}q^\nu$.

Returning to 9, we then see that the first term contains γ_μ and $\gamma_\mu\gamma_5$ but is independent of q_μ. The second term contains the $\sigma_{\mu\nu}q^\nu$ term from $(p+p')_\mu$, and thus is proportional to $\sigma_{\mu\nu}$ (although I'm not quite sure why there is an extra term $\tilde{F}_2\gamma_5$ there, since it doesn't appear as one of the basis matrices). The third term contains $\gamma_\mu\gamma_5$ terms in the factor $\not{q}\gamma_5$, but now the vector index is attached to q_μ. The final term contains the identity matrix 1 (multiplied by $q_\mu F_4$) and a single γ_5, also multiplied by $q_\mu\tilde{F}_4$. Thus we can see that all 5 basis matrices from the set 10 are there, and combined with q_μ such that each term has exactly one vector index μ.

L&P also note that although the first, second and fourth terms in 9 come in pairs, one with a lone F matrix and one with an F matrix multiplied by γ_5, there is no term in the third term consisting of F_3 on its own. They state that this is because $\overline{u}(\mathbf{p}')\not{q}u(\mathbf{p})=0$, which can be seen as follows, using 12 and 11.

$$\overline{u}\left(\mathbf{p}'\right)\not{q}u(\mathbf{p}) = \overline{u}\left(\mathbf{p}'\right)q_\mu\gamma^\mu u(\mathbf{p}) \tag{13}$$
$$= \frac{1}{2m}\overline{u}\left(\mathbf{p}'\right)q_\mu\left[\left(p+p'\right)^\mu - i\sigma^{\mu\alpha}q_\alpha\right]u(\mathbf{p}) \tag{14}$$
$$= \frac{1}{2m}\overline{u}\left(\mathbf{p}'\right)\left[\left(p-p'\right)_\mu\left(p+p'\right)^\mu - i\sigma^{\mu\alpha}q_\alpha q_\mu\right]u(\mathbf{p}) \tag{15}$$
$$= \frac{1}{2m}\overline{u}\left(\mathbf{p}'\right)\left[p^2-p'^2+\frac{1}{2}\left[\gamma^\mu,\gamma^\alpha\right]q_\alpha q_\mu\right]u(\mathbf{p}) \tag{16}$$

Since the momenta p and p' refer to the same particle, we have

$$p^2-p'^2 = m^2-m^2 = 0 \tag{17}$$

For the last term in 16 we have

$$\left[\gamma^\mu,\gamma^\alpha\right]q_\alpha q_\mu = \left(\gamma^\mu\gamma^\alpha-\gamma^\alpha\gamma^\mu\right)q_\alpha q_\mu \tag{18}$$
$$= \not{q}^2-\not{q}^2 \tag{19}$$
$$= 0 \tag{20}$$

Therefore

$$\overline{u}\left(\mathbf{p}'\right)\not{q}u(\mathbf{p}) = 0 \tag{21}$$

Again, I'm not sure if there is a way to derive 9 step-by-step, but we can (sort of) see that it works.

L&P then use the fact that current is conserved, which means that

$$\partial^{\mu} j_{\mu}(x) = 0 \tag{22}$$

Applying this to 8, we see that the only place where x appears is in the exponential $e^{-iq\cdot x}$, so we get

$$\left\langle \mathbf{p}', s' \left| \partial^{\mu} j_{\mu}(x) \right| \mathbf{p}, s \right\rangle = -iq^{\mu} \left\langle \mathbf{p}', s' \left| j_{\mu}(x) \right| \mathbf{p}, s \right\rangle = 0 \tag{23}$$

We can then apply this condition to 9 and L&P show that this leads to the form

$$\Gamma_{\mu} = \gamma_{\mu} F_1 + \left(iF_2 + \tilde{F}_2 \gamma_5 \right) \sigma_{\mu\nu} q^{\nu} + \tilde{F}_3 \left(q_{\mu} \not{q} - q^2 \gamma_{\mu} \right) \gamma_5 \tag{24}$$

Lorentz invariance means that the F matrices can depend only on scalar combinations of the momenta p and p', which are p^2, p'^2 and $p \cdot p'$. The first two are both just equal to m^2 so we're left with $p \cdot p'$. We can express the dependence on $p \cdot p'$ just as well by a dependence on $q^2 = p^2 + p'^2 - 2p \cdot p'$ so we therefore have that all the F matrices can be functions of at most q^2 (although they could also be constants). The matrices F are known as *form factors*.

In problem 11.1, L&P ask us to work out the general matrix element for an electromagnetic current between two complex scalar states. Again, I'm not entirely sure of what is meant here, but presumably we can work somewhat in analogy to what was done for the Dirac fields above. Going back to the definition of scalar states, we have

$$\phi_{+}(x) \left| B(\mathbf{k}) \right\rangle = \frac{1}{\sqrt{2E_k V}} e^{-ik\cdot x} \left| 0 \right\rangle \tag{25}$$

$$\left\langle B(\mathbf{k}) \right| \phi_{-}(x) = \frac{1}{\sqrt{2E_k V}} e^{ik\cdot x} \left\langle 0 \right| \tag{26}$$

I'm guessing that the analog of 8 for the scalar case is

$$\left\langle B\left(\mathbf{p}'\right) \left| j_{\mu}(x) \right| B(\mathbf{p}) \right\rangle = \frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\sqrt{2E_{p'} V}} e\Gamma_{\mu}\left(p, p'\right) \tag{27}$$

In this case, Γ_{μ} is a scalar, not a 4×4 matrix, since the Dirac spinors are now absent. As before, however, Γ_{μ} must depend on a single vector index μ, which now must come from the momenta p_{μ} and p'_{μ}, since there are no gamma matrices any more. Applying the conservation of current condition 22 leads to

$$q^{\mu} \Gamma_{\mu} = 0 \tag{28}$$

The only combination of p_{μ} and p'_{μ} that satisfies this condition is $(p + p')_{\mu}$ since

$$q^{\mu}\left(p+p'\right)_{\mu}=\left(p-p'\right)^{\mu}\left(p+p'\right)_{\mu}=p^{2}-p'^{2}=0 \tag{29}$$

As before, Lorentz invariance means that the scalar part of Γ_{μ} can depend only on q^{2}, so we get for the most general form

$$\Gamma_{\mu}=\left(p+p'\right)_{\mu}F\left(q^{2}\right) \tag{30}$$

where F is now a scalar function (not a matrix) and we've left the elementary charge in as an explicit factor since it always appears in an electromagnetic vertex. Inserting this back into 27 we have

$$\left\langle B\left(\mathbf{p}'\right)\left|j_{\mu}\left(x\right)\right|B\left(\mathbf{p}\right)\right\rangle =\frac{e^{-iq\cdot x}}{\sqrt{2E_{p}V}\sqrt{2E_{p'}V}}\left(p+p'\right)_{\mu}eF\left(q^{2}\right) \tag{31}$$

As I stated at the start, I'm not really happy with my understanding of the concepts underlying all this, as a lot of it seems to be hand-waving. Comments welcome, as always.

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