DISCRETE FOURIER TRANSFORMS

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The regular Fourier transform of a function $f(t)$ is defined by

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

where ω is the angular frequency if $f(t)$ is a function of time. In any real-life experiment, of course, we will get only a finite number of measurements of f, so we can't compute the Fourier transform exactly. In such situations, we can use the *discrete Fourier transform* to get an estimate of the frequency spectrum $F(\omega)$. The idea is that we make a series of N measurements at equally spaced intervals of T per interval, with the first measurement at $t = 0$ and the last at $t = (N-1)T$. We then assume that the function we're sampling is periodic, repeating its pattern over time intervals outside those in which we made measurements. That is, the values of f for $t = Nt$ up to $t = (2N-1)t$ repeat those from $t = 0$ to $t = (N-1)t$. In that case, we can calculate the Fourier transform by integrating over the single cycle for which we have data:

$$F(\omega) = \int_{0}^{(N-1)T} f(t) e^{-i\omega t} dt \tag{2}$$

Of course, we still have data only for the N discrete times at which we made measurements, so we now make the approximation that f is constant over each interval T. The transform then becomes

$$F(\omega) \approx \sum_{k=0}^{N-1} f_k e^{-i\omega kT} \tag{3}$$

Since we're treating the data as periodic with period NT, we can look for frequencies equal to $\frac{1}{NT}$ Hz $= \frac{2\pi}{NT}$ s^{-1} and higher harmonics (multiples of the fundamental frequency). We could also have a constant component in the data, so we'll include $\omega = 0$ as well. That is, we work out the sum 3 for all values of $\omega = 0, \frac{2\pi}{NT}, 2\frac{2\pi}{NT}, 3\frac{2\pi}{NT}, \ldots, (N-1)\frac{2\pi}{NT}$. The final formula for the discrete Fourier transform is thus

$$\boxed{F_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi nk/N}} \tag{4}$$

for $n = 0, 1, \ldots, N-1$.

The inverse transform is given by

$$\boxed{f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{i2\pi nk/N}} \tag{5}$$

To see this, substitute 5 back into 4 for F_m:

$$F_m = \sum_{k=0}^{N-1} f_k e^{-i2\pi mk/N} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} F_n e^{i2\pi nk/N} e^{-i2\pi mk/N} \tag{6}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} F_n \sum_{k=0}^{N-1} e^{i2\pi(n-m)k/N} \tag{7}$$

The sum over k is N if $n = m$. If $n \neq m$ we have a geometric series:

$$\sum_{k=0}^{N-1} e^{i2\pi(n-m)k/N} = \sum_{k=0}^{N-1} x^k \tag{8}$$

where

$$x \equiv e^{i2\pi(n-m)/N} \tag{9}$$

The sum of a geometric series (found in most calculus textbooks) is

$$\sum_{k=0}^{N-1} x^k = \frac{1-x^N}{1-x} \tag{10}$$

so we get

$$\sum_{k=0}^{N-1} e^{i2\pi(n-m)k/N} = \frac{1-e^{i2\pi(n-m)}}{1-e^{i2\pi(n-m)/N}} \tag{11}$$

$$= 0 \tag{12}$$

since $n-m$ is an integer, so $e^{i2\pi(n-m)} = 1$ in the numerator. Therefore

$$\sum_{k=0}^{N-1} e^{i2\pi(n-m)k/N} = N\delta_{nm} \tag{13}$$

Returning to 7 we get

$$\sum_{k=0}^{N-1} f_k e^{-i2\pi nk/N} = \sum_{n=0}^{N-1} F_n \delta_{nm} = F_m \tag{14}$$

which shows that 5 is indeed the inverse transform.

The transform 4 can be written as a matrix equation. For example, if $N = 4$, we have

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{15}$$

where

$$W \equiv e^{-i2\pi/N} \tag{16}$$

The matrix is symmetric, but we can also decrease the number of matrix elements we need to calculate by noting that $W^N = e^{-i2\pi} = 1$, so $W^6 = W^2$ and $W^9 = W^5 = W$:

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{17}$$

In most applications, the original data f_k are real, which allows us to derive a relation among the transform elements F_n.

$$F_{N-n} = \sum_{k=0}^{N-1} f_k e^{-i2\pi(N-n)k/N} \tag{18}$$

$$= \sum_{k=0}^{N-1} f_k e^{-i2\pi k} e^{i2\pi nk/N} \tag{19}$$

$$= \sum_{k=0}^{N-1} f_k e^{i2\pi nk/N} \tag{20}$$

$$= F_n^* \tag{21}$$

where in line 2, since k is an integer, $e^{-i2\pi k} = 1$. Therefore, both F_{N-n} and F_n contribute to the same frequency component n. From 5 we can get this frequency component $f_{k,n}$:

$$f_{k,n} = \frac{1}{N}\left[F_n e^{i2\pi nk/N} + F_n^* e^{i2\pi(N-n)k/N}\right] \tag{22}$$

$$= \frac{1}{N}\left[F_n e^{i2\pi nk/N} + F_n^* e^{-i2\pi nk/N}\right] \tag{23}$$

The second term in the sum is the complex conjugate of the first (which it would have to be, if $f_{k,n}$ is to be real). We can write the first term in modulus-argument form as

$$F_n e^{i2\pi nk/N} = |F_n| e^{i(2\pi nk/N+\phi)} \tag{24}$$

where ϕ is the argument of F_n. Therefore

$$f_{k,n} = 2\frac{|F_n|}{N}\cos\left(\frac{2\pi nk}{N}+\phi\right) \tag{25}$$

$$= 2\frac{|F_n|}{N}\cos\left(\frac{2\pi n}{NT}kT+\phi\right) \tag{26}$$

The original data point f_k is the measurement taken at $t = kT$, so we can see that the contribution from component F_n is a sine wave with frequency $\omega = \frac{2\pi n}{NT}$, phase ϕ and amplitude $2\frac{|F_n|}{N}$.

There are a couple of special cases. First, if $n = N - n = \frac{N}{2}$ then there is only one frequency component that contributes so

$$f_{k,\frac{N}{2}} = \frac{|F_n|}{N}\cos\left(\frac{2\pi n}{NT}kT+\phi\right) \tag{27}$$

Second, for $n=0$ we get from 5

$$f_{k,0}=\frac{F_0}{N} \tag{28}$$

Example 1. Suppose the data are drawn from the function

$$f(t)=3+\cos(\pi t)+\sin(\pi t) \tag{29}$$

Taking $T=\frac{1}{2}$ we get

k	f_k
0	4
1	4
2	2
3	2

We get

$$W=e^{-i\pi/2}=-i \tag{30}$$

The transform is

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 2 \\ 2 \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} 12 \\ 2-2i \\ 0 \\ 2+2i \end{bmatrix} \tag{32}$$

The constant component is

$$\frac{F_0}{N}=3 \tag{33}$$

For $n=1$, $\omega=2\pi n/NT=\pi$ and

$$\phi = \arg(2-2i)=-\frac{\pi}{4} \tag{34}$$

$$f_{k,1} = 2\frac{|F_1|}{N} \tag{35}$$

$$= \sqrt{2} \tag{36}$$

So the component with $\omega=\pi$ is

$$\sqrt{2}\cos\left(\pi t-\frac{\pi}{4}\right) = \sqrt{2}\left(\cos\pi t\cos\frac{\pi}{4}+\sin\pi t\sin\frac{\pi}{4}\right) \quad (37)$$

$$= \cos(\pi t)+\sin(\pi t) \quad (38)$$

So we do actually get back the exact original function.

Example 2. As you might expect, if the sampling interval gets close to (or, worse, exceeds) the actual periods in the data, the method breaks down. If we tried

$$f(t)=3+\cos(2\pi t)+\sin(4\pi t) \quad (39)$$

but kept the sampling interval the same, at $T=\frac{1}{2}$, then we get

k	f_k
0	4
1	2
2	4
3	2

The transform is

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} \quad (40)$$

$$= \begin{bmatrix} 12 \\ 0 \\ 4 \\ 0 \end{bmatrix} \quad (41)$$

The constant component is

$$\frac{F_0}{N}=3 \quad (42)$$

so that's still ok. For $n=1$, $\omega=2\pi n/NT=\pi$ and

$$\phi = \arg(0)=0 \quad (43)$$

$$f_{k,1} = 2\frac{|F_1|}{N} \quad (44)$$

$$= 0 \quad (45)$$

so there is no component with $\omega=\pi$, which is also ok.

For $n=2$, we get from 27

$$\begin{aligned}
\phi &= \arg\left(F_2\right)=0 && (46)\\
f_{k,2} &= \frac{|F_2|}{N} && (47)\\
&= 1 && (48)
\end{aligned}$$

so we reclaim the $\cos\left(2\omega t\right)$ term. However, there is no way to recover the $\sin\left(4\omega t\right)$ term since our sampling intervals all hit values of t where $\sin\left(4\omega t\right)=0$, so it's as if this term didn't exist.

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