

LAGRANGIANS AND THE PRINCIPLE OF LEAST ACTION

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Section 1.4.

A fundamental application of functional derivatives is in the derivation of the principle of least action and the Euler-Lagrange equation. A particle with mass m following some trajectory $x(t)$ will have a potential energy $V(x)$ and a kinetic energy $T(x)$ at each point on the trajectory. If the particle follows the trajectory between times $t = 0$ and $t = \tau$, then these energies have averages given as functionals

$$(1) \quad \bar{V}[x] = \frac{1}{\tau} \int_0^\tau V(x(t)) dt$$

$$(2) \quad \bar{T}[x] = \frac{1}{\tau} \int_0^\tau T(x(t)) dt$$

$$(3) \quad = \frac{m}{2\tau} \int_0^\tau \dot{x}^2 dt$$

From example 1 in the earlier post, we have

$$(4) \quad \frac{\delta \bar{V}[x]}{\delta x(t)} = \frac{1}{\tau} \frac{\partial V}{\partial x}$$

From example 2 in the same post, we have

$$(5) \quad \frac{\delta \bar{T}[x]}{\delta x(t)} = \frac{1}{\tau} \left(\frac{\partial T}{\partial x} - \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] \right)$$

$$(6) \quad = \frac{1}{\tau} (0 - m\ddot{x})$$

$$(7) \quad = -\frac{m\ddot{x}}{\tau}$$

In classical physics, all forces are ultimately represented by conservative forces (gravity and electromagnetism), so a force can be represented by the gradient of a potential: $F = -\frac{\partial V}{\partial x}$. Newton's law $F = ma$ then says

$$(8) \quad m\ddot{x} = -\frac{\partial V}{\partial x}$$

so in terms of functional derivatives

$$(9) \quad \frac{\delta \bar{T}[x]}{\delta x(t)} = \frac{\delta \bar{V}[x]}{\delta x(t)}$$

That is, the variation in the average kinetic energy over the path is equal to the variation in the average potential energy. This suggests that the quantity

$$(10) \quad L \equiv T - V$$

known as the *Lagrangian*, is in some sense fundamental, since the variation in its integral over a path is zero. To give a name to this integral, we define the *action* S as

$$(11) \quad S \equiv \int_0^\tau L dt$$

giving rise to the condition

$$(12) \quad \frac{\delta S[x]}{\delta x(t)} = 0$$

Technically, this means that the action is stationary when the trajectory $x(t)$ is the actual trajectory followed by a particle. In practice, the action nearly always turns out to be a minimum, so this is known as the *principle of least action*. Varying the trajectory slightly causes the action to increase and gives a path that is not followed by the particle.

If we know the Lagrangian for a particle (or more generally, for a system of particles), we can work out the equation(s) of motion by applying the principle of least action. To do this, we consider the Lagrangian as a function of a generalized coordinate $x(t)$ and its first derivative $\dot{x}(t)$:

$$(13) \quad L = L(x, \dot{x})$$

We can then apply the principle of least action:

$$(14) \quad \frac{\delta S[x]}{\delta x(t)} = \frac{\delta}{\delta x(t)} \left[\int_0^\tau L(x, \dot{x}) dt \right] = 0$$

This has the same form as example 2 from the earlier post that we used above, so we get

$$(15) \quad \frac{\delta}{\delta x(t)} \left[\int_0^\tau L(x, \dot{x}) dt \right] = \frac{\partial L}{\partial x} - \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}} \right] = 0$$

Thus we get the *Euler-Lagrange equation* for a single particle moving in one dimension

$$(16) \quad \boxed{\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0}$$

Example. As a simple example, suppose we have a particle whose potential and kinetic energies are given by

$$(17) \quad V(x, \dot{x}) = \frac{1}{2} kx^2$$

$$(18) \quad T(x, \dot{x}) = \frac{1}{2} m\dot{x}^2$$

Then the equation of motion is given by

$$(19) \quad L(x, \dot{x}) = \frac{1}{2} (m\dot{x}^2 - kx^2)$$

$$(20) \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -kx - m\ddot{x} = 0$$

$$(21) \quad m\ddot{x} = -kx$$

This is just the equation of motion for a mass on a spring with spring constant k .

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