LAGRANGIANS AND THE PRINCIPLE OF LEAST ACTION

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Section 1.4.

A fundamental application of functional derivatives is in the derivation of the principle of least action and the Euler-Lagrange equation. A particle with mass m following some trajectory $x(t)$ will have a potential energy $V(x)$ and a kinetic energy $T(x)$ at each point on the trajectory. If the particle follows the trajectory between times $t=0$ and $t=\tau$, then these energies have averages given as functionals

$$\bar{V}[x] = \frac{1}{\tau}\int_0^{\tau} V(x(t))\,dt \tag{1}$$

$$\bar{T}[x] = \frac{1}{\tau}\int_0^{\tau} T(x(t))\,dt \tag{2}$$

$$= \frac{m}{2\tau}\int_0^{\tau} \dot{x}^2 dt \tag{3}$$

From example 1 in the earlier post, we have

$$\frac{\delta\bar{V}[x]}{\delta x(t)} = \frac{1}{\tau}\frac{\partial V}{\partial x} \tag{4}$$

From example 2 in the same post, we have

$$\frac{\delta\bar{T}[x]}{\delta x(t)} = \frac{1}{\tau}\left(\frac{\partial T}{\partial x} - \frac{d}{dt}\left[\frac{\partial T}{\partial\dot{x}}\right]\right) \tag{5}$$

$$= \frac{1}{\tau}(0 - m\ddot{x}) \tag{6}$$

$$= -\frac{m\ddot{x}}{\tau} \tag{7}$$

In classical physics, all forces are ultimately represented by conservative forces (gravity and electromagnetism), so a force can be represented by the gradient of a potential: $F=-\frac{\partial V}{\partial x}$. Newton's law $F=ma$ then says

$$m\ddot{x} = -\frac{\partial V}{\partial x} \tag{8}$$

so in terms of functional derivatives

$$\frac{\delta \bar{T}[x]}{\delta x(t)} = \frac{\delta \bar{V}[x]}{\delta x(t)} \tag{9}$$

That is, the variation in the average kinetic energy over the path is equal to the variation in the average potential energy. This suggests that the quantity

$$L \equiv T - V \tag{10}$$

known as the *Lagrangian*, is in some sense fundamental, since the variation in its integral over a path is zero. To give a name to this integral, we define the *action* S as

$$S \equiv \int_0^\tau L dt \tag{11}$$

giving rise to the condition

$$\frac{\delta S[x]}{\delta x(t)} = 0 \tag{12}$$

Technically, this means that the action is stationary when the trajectory $x(t)$ is the actual trajectory followed by a particle. In pracitice, the action nearly always turns out to be a minimum, so this is known as the *principle of least action*. Varying the trajectory slightly causes the action to increase and gives a path that is not followed by the particle.

If we know the Lagrangian for a particle (or more generally, for a system of particles), we can work out the equation(s) of motion by applying the principle of least action. To do this, we consider the Lagrangian as a function of a generalized coordinate $x(t)$ and its first derivative $\dot{x}(t)$:

$$L = L(x, \dot{x}) \tag{13}$$

We can then apply the principle of least action:

$$\frac{\delta S[x]}{\delta x(t)} = \frac{\delta}{\delta x(t)} \left[\int_0^\tau L(x, \dot{x})\, dt \right] = 0 \tag{14}$$

This has the same form as example 2 from the earlier post that we used above, so we get

$$\frac{\delta}{\delta x(t)}\left[\int_0^{\tau} L(x,\dot{x})\,dt\right]=\frac{\partial L}{\partial x}-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right]=0 \tag{15}$$

Thus we get the *Euler-Lagrange equation* for a single particle moving in one dimension

$$\boxed{\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0} \tag{16}$$

Example. As a simple example, suppose we have a particle whose potential and kinetic energies are given by

$$V(x,\dot{x}) = \frac{1}{2}kx^2 \tag{17}$$

$$T(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 \tag{18}$$

Then the equation of motion is given by

$$L(x,\dot{x}) = \frac{1}{2}\left(m\dot{x}^2-kx^2\right) \tag{19}$$

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -kx-m\ddot{x}=0 \tag{20}$$

$$m\ddot{x} = -kx \tag{21}$$

This is just the equation of motion for a mass on a spring with spring constant k.

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