FERMAT'S PRINCIPLE OF LEAST TIME AND SNELL'S LAW

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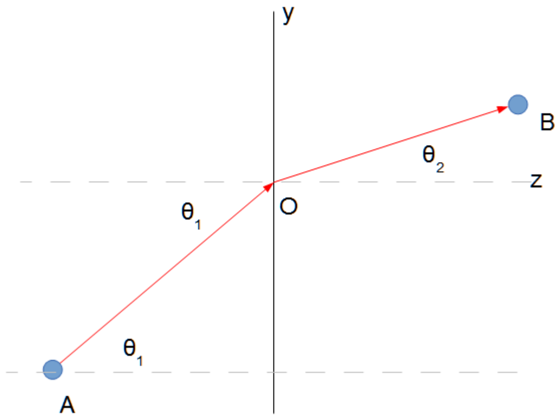
References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 1.1

Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Prologue, problem 1.

One of the guiding principles of quantum field theory is that a particle travelling between two points actually traverses all possible paths between these two points, although with varying probabilities for different paths. Although this idea is expressed mathematically using the calculus of variations, a simpler example of the same idea is that of Fermat's principle of least time applied to the derivation of Snell's law of refraction in optics.

The idea is that given that the speed of light in a medium with index of refraction n is c/n, if a light beam starts at a point A in medium 1 and hits the interface between mediums 1 and 2 at an angle θ_1 to the normal, and continues through into medium 2 at an angle θ_2 to the normal eventually arriving at point B, then these angles are such that the travel time from A to B is a minimum. There isn't any particular reason why this assumption is made (apart from the the fact that it gives the right answer!).

To see how it works, suppose we orient the interface so that it lies in the xy plane, so that the normal to the interface is the z axis. We'll take the incident beam of light starting at point A to lie in the yz plane, as does the refracted beam which travels from the interface to point B. We'll let y_{AB} be the difference in y coordinate of the points A and B, and let y_{AO} be the difference in y coordinate of the point O where the beam hits the interface. Thus the difference in y coordinate between O and B is $y_{OB} = y_{AB} - y_{AO}$. Similarly, let z_{AO} and z_{OB} be the differences in z coordinates between the corresponding points. Finally, let a be the distance from A to O, and b the distance from O to B.



Then by Pythagoras

$$a = \sqrt{z_{AO}^2 + y_{AO}^2} \tag{1}$$

$$b = \sqrt{z_{OB}^2 + (y_{AB} - y_{AO})^2} \tag{2}$$

The total travel time of the light beam is

$$t = \frac{a}{v_1} + \frac{b}{v_2} \tag{3}$$

$$= \frac{a}{c} n_1 + \frac{b}{c} n_2 \tag{4}$$

$$ct = n_1 \sqrt{z_{AO}^2 + y_{AO}^2} + n_2 \sqrt{z_{OB}^2 + (y_{AB} - y_{AO})^2} \tag{5}$$

Since the points A and B are fixed, as is the location of the interface, the only thing we can vary is y coordinate of the point where the light beam hits the interface, that is, y_{AO}. We can therefore minimize ct with respect to y_{AO}:

$$\frac{d\,(ct)}{dy_{AO}} = \frac{y_{AO} n_1}{\sqrt{z_{AO}^2 + y_{AO}^2}} - \frac{n_2\,(y_{AB} - y_{AO})}{\sqrt{z_{OB}^2 + (y_{AB} - y_{AO})^2}} = 0 \tag{6}$$

$$\frac{y_{AO}}{\sqrt{z_{AO}^2 + y_{AO}^2}} n_1 = \frac{(y_{AB} - y_{AO})}{\sqrt{z_{OB}^2 + (y_{AB} - y_{AO})^2}} n_2 \tag{7}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{8}$$

where the last line uses the trigonometric definition of the sine from the sides of the triangles.