FUNCTIONAL DERIVATIVES: MORE EXAMPLES

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problems 1.3, 1.4.

Here are a few more examples of functional derivatives.

Example 1. Let the functional $G[f]$ be

$$G[f]=\int g(y,f)\,dy \tag{1}$$

[We'll assume that all integrals in these examples have limits, but to ease the writing, I'll leave them off.] Then we get for the functional derivative

$$\frac{\delta G[f]}{\delta f(x)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[g(y,f+\varepsilon\delta(y-x))-g(y,f)\right]dy \tag{2}$$

Since we're interested only in terms in the integrand that are first order in ε, we can Taylor-expand $g(y,f+\varepsilon\delta(y-x))$ to first order, and we get

$$\frac{\delta G[f]}{\delta f(x)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[g(y,f)+\varepsilon\frac{\partial g(y,f)}{\partial f}\delta(y-x)-g(y,f)\right]dy \tag{3}$$

$$=\int\frac{\partial g(y,f)}{\partial f}\delta(y-x)\,dy \tag{4}$$

$$=\frac{\partial g(x,f)}{\partial f} \tag{5}$$

[We're also assuming that in integrals of the form $\int f(y)\,\delta(y-x)\,dy$ that x is always within the limits of integration so that the result is $\int f(y)\,\delta(y-x)\,dy=f(x)$. In all cases, if x is outside the limits of integration, then the integral evaluates to zero.]

Example 2. Extending the previous example, suppose we now have

$$H[f]=\int g\left(y,f,f'\right)dy \tag{6}$$

where $f'=\frac{df}{dy}$. Then

(7)
$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[g\left(y, f+\varepsilon\delta(y-x), f'+\varepsilon\delta'(y-x)\right) - g\left(y,f,f'\right)\right]dy$$

Treating f and f' as independent variables, we can Taylor expand to first order in ε and take the limit:

$$\frac{\delta H[f]}{\delta f(x)} = \int\left[\frac{\partial g(y,f,f')}{\partial f}\delta(y-x) + \frac{\partial g(y,f,f')}{\partial f'}\delta'(y-x)\right]dy \tag{8}$$

The first term is the same as in example 1. We can integrate the second term by parts to get

(9)
$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g(x,f,f')}{\partial f} + \int\frac{\partial g(y,f,f')}{\partial f'}\delta'(y-x)\,dy$$
(10)
$$= \frac{\partial g(x,f,f')}{\partial f} + \frac{\partial g(y,f,f')}{\partial f'}\delta(y-x) - \int\frac{d}{dy}\left[\frac{\partial g(y,f,f')}{\partial f'}\right]\delta(y-x)\,dy$$
(11)
$$= \frac{\partial g(x,f,f')}{\partial f} - \frac{d}{dx}\left[\frac{\partial g(x,f,f')}{\partial f'}\right]$$

In the second line, we assume that x isn't one of the limits of integration, so the second term is zero at both limits.

Example 3. We can extend this example by one more step (by which time you should see a pattern forming!) with

$$J[f] = \int g\left(y,f,f',f''\right)dy \tag{12}$$

To find $\frac{\delta J[f]}{\delta f(x)}$ we can treat f, f' and f'' as independent variables and so we get

(13)
$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[g\left(y, f+\varepsilon\delta(y-x), f'+\varepsilon\delta'(y-x), f''+\varepsilon\delta''(y-x)\right) - g\left(y,f,f',f''\right)\right]dy$$

By Taylor-expanding with respect to all three variables, we just repeat example 2 for the first 2 terms. For the third term, we get

$$\int \frac{\partial g}{\partial f''}\delta''(y-x)\,dy = \frac{\partial g}{\partial f''}\delta'(y-x) - \int \frac{d}{dy}\frac{\partial g}{\partial f''}\delta'(y-x)\,dy \tag{14}$$

$$= -\frac{d}{dy}\frac{\partial g}{\partial f''}\delta(y-x) + \int \frac{d^2}{dy^2}\frac{\partial g}{\partial f''}\delta(y-x)\,dy \tag{15}$$

$$= \frac{d^2}{dx^2}\frac{\partial g}{\partial f''} \tag{16}$$

Putting it all together, we have

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx}\left[\frac{\partial g}{\partial f'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial g}{\partial f''}\right] \tag{17}$$

Example 4. The functional derivative of a function with respect to itself is

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\phi(x) + \varepsilon\delta(x-y) - \phi(x)\right] \tag{18}$$

$$= \delta(x-y) \tag{19}$$

This is the analog of the ordinary derivative

$$\frac{dy}{dx} = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

That is, if y and x are the same independent variable then the derivative is 1, but if they are different *independent* variables (that is, y isn't a function of x), then the derivative is zero, since the two variables are, well, independent.

Example 5. By extension of example 4, we have (where a dot above a symbol indicates a time derivative)

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \frac{d}{dt}\frac{\delta\phi(t)}{\delta\phi(t_0)} \tag{21}$$

$$= \frac{d}{dt}\delta(t-t_0) \tag{22}$$

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