LAGRANGIANS FOR ELASTIC MEDIA

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 1.5.

As another application of the principle of least action we will look at a vibrating string. Suppose we have a string stretched between two points so that its tension is $\mathcal{T}$. If we pluck the string so that it starts vibrating, then at a point x on the line connecting the points to which the ends of the string are joined, the displacement from equilibrium of a point on the string is given by $\psi(x,t)$. Assuming that a point on the string can move only up and down (that is, perpendicular to the line joining the ends), then the kinetic energy of a segment of length dx is (given that the string's linear density is ρ):

$$dT = \frac{1}{2}\rho\, dx\, \dot{\psi}^2 \tag{1}$$

To get the potential energy, we can use the following argument. If a segment of length dx is displaced from equilibrium by a distance $d\psi$ by the action of the tension $\mathcal{T}$, then the stretched length of that segment is

$$ds = \sqrt{dx^2 + d\psi^2} \approx dx\left(1 + \frac{1}{2}\left(\frac{d\psi}{dx}\right)^2\right) \tag{2}$$

to first order in dx. The work done to stretch the string is the force $\mathcal{T}$ times the distance stretched, which is $ds - dx$ so the work done, which is equal to the potential energy, is

$$dV = \mathcal{T}(ds - dx) = \frac{\mathcal{T}}{2}\left(\frac{d\psi}{dx}\right)^2 dx \tag{3}$$

We can now introduce the *Lagrangian density* (effectively, the Lagrangian per unit length):

$$\mathcal{L} \equiv \frac{dL}{dx} = \frac{dT - dV}{dx} = \frac{\rho}{2}(\partial_t\psi)^2 - \frac{\mathcal{T}}{2}(\partial_x\psi)^2 \tag{4}$$

The action can now be written as

$$S = \int \mathcal{L}\, dx\, dt \tag{5}$$

To minimize the action we now have to consider a functional that depends on a function of two independent variables. That is, we have

$$\mathcal{L} = \mathcal{L}\left(\psi, \partial_x \psi, \partial_t \psi\right) \tag{6}$$

where

$$\psi = \psi\left(x, t\right) \tag{7}$$

We can generalize the original definition of a functional derivative by considering a variation of the function ψ at one point (x_0, t_0), that is

$$\psi\left(x, t\right) \rightarrow \psi\left(x, t\right) + \varepsilon \delta\left(x - x_0\right) \delta\left(t - t_0\right) \tag{8}$$

To see how this works, consider the slightly simpler case where the function $\mathcal{L}$ depends only on ψ and not on its derivatives. Then

$$\frac{\delta S\left[\psi\left(x, t\right)\right]}{\delta \psi\left(x_0, t_0\right)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int dt \int dx \left[\mathcal{L}\left(\psi + \varepsilon \delta\left(x - x_0\right) \delta\left(t - t_0\right)\right) - \mathcal{L}\left(\psi\right)\right] \tag{9}$$

$$= \int dt \int dx \frac{\partial \mathcal{L}}{\partial \psi\left(x, t\right)} \delta\left(x - x_0\right) \delta\left(t - t_0\right) \tag{10}$$

$$= \int dt \frac{\partial \mathcal{L}}{\partial \psi\left(x_0, t\right)} \delta\left(t - t_0\right) \tag{11}$$

$$= \frac{\partial \mathcal{L}}{\partial \psi\left(x_0, t_0\right)} \tag{12}$$

where we used a Taylor expansion of the first line up to first order to get the second line.

To extend this to the more general case where $\mathcal{L} = \mathcal{L}\left(\psi, \partial_x \psi, \partial_t \psi\right)$ we can follow example 2 in this earlier post to get

$$\frac{\delta S\left[\psi\left(x, t\right)\right]}{\delta \psi\left(x_0, t_0\right)} = \frac{\partial \mathcal{L}}{\partial \psi\left(x_0, t_0\right)} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \left(\partial_x \psi\left(x_0, t_0\right)\right)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \left(\partial_t \psi\left(x_0, t_0\right)\right)} = 0 \tag{13}$$

Applying this to 4 we get

$$\frac{\partial \mathcal{L}}{\partial \psi\left(x_0, t_0\right)} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \left(\partial_x \psi\left(x_0, t_0\right)\right)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \left(\partial_t \psi\left(x_0, t_0\right)\right)} = 0 + \mathcal{T} \frac{d}{dx}\left(\partial_x \psi\right) - \rho \frac{d}{dt}\left(\partial_t \psi\right) = 0 \tag{14}$$

$$\partial_x^2 \psi = \frac{\rho}{\mathcal{T}} \partial_t^2 \psi \tag{15}$$

The final equation is just the wave equation that we derived earlier when studying electromagnetic waves.

We can generalize this to a three-dimensionsal elastic medium by taking $\psi(x,y,z,t) = \psi(\mathbf{r},t)$ to be a three-dimensional scalar field (actually, it's not clear exactly what ψ represents in this case; in the 'real' theory of 3-d elasticity, the displacement of an element of the elastic medium is a vector field, not a scalar field, as you would expect). We can generalize the idea of a Lagrangian density to three dimensions, so we have

$$\mathcal{L} = \frac{\rho}{2}(\partial_t \psi)^2 - \frac{\mathcal{T}}{2}(\nabla \psi)^2 \tag{16}$$

The principle of least action is now

$$\frac{\delta S[\psi(\mathbf{r},t)]}{\delta \psi(\mathbf{r}_0,t_0)} = \frac{\delta}{\delta \psi(\mathbf{r}_0,t_0)} \int dt \int d^3\mathbf{r} \left[\frac{\rho}{2}(\partial_t \psi)^2 - \frac{\mathcal{T}}{2}(\nabla \psi)^2 \right] \tag{17}$$

$$= \frac{\delta}{\delta \psi(\mathbf{r}_0,t_0)} \int dt \int d^3\mathbf{r} \left[\frac{\rho}{2}(\partial_t \psi)^2 - \frac{\mathcal{T}}{2}\left[(\partial_x \psi)^2 + (\partial_y \psi)^2 + (\partial_z \psi)^2 \right] \right] \tag{18}$$

ψ is now a function of 4 independent variables, so we can use the same argument as above to calculate the functional derivative. We perturb ψ by the amount $\varepsilon \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t-t_0)$ and follow through the derivation in the same way. All the terms in the integral are of the form we looked at in example 4 in this post, so we get

$$\frac{\delta S[\psi(\mathbf{r},t)]}{\delta \psi(\mathbf{r}_0,t_0)} = -\rho \partial_t^2 \psi + \mathcal{T}\left(\partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi \right) = 0 \tag{19}$$

$$\nabla^2 \psi = \frac{\rho}{\mathcal{T}}\partial_t^2 \psi \tag{20}$$

We get the 3-d wave equation.

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