LAGRANGIANS FOR ELASTIC MEDIA

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 1.5.

As another application of the principle of least action we will look at a vibrating string. Suppose we have a string stretched between two points so that its tension is $\mathcal{T}$. If we pluck the string so that it starts vibrating, then at a point x on the line connecting the points to which the ends of the string are joined, the displacement from equilibrium of a point on the string is given by $\psi(x,t)$. Assuming that a point on the string can move only up and down (that is, perpendicular to the line joining the ends), then the kinetic energy of a segment of length dx is (given that the string's linear density is ρ):

$$dT=\frac{1}{2}\rho dx\dot{\psi}^2 \tag{1}$$

To get the potential energy, we can use the following argument. If a segment of length dx is displaced from equilibrium by a distance $d\psi$ by the action of the tension $\mathcal{T}$, then the stretched length of that segment is

$$ds=\sqrt{dx^2+d\psi^2}\approx dx\left(1+\frac{1}{2}\left(\frac{d\psi}{dx}\right)^2\right) \tag{2}$$

to first order in dx. The work done to stretch the string is the force $\mathcal{T}$ times the distance stretched, which is $ds-dx$ so the work done, which is equal to the potential energy, is

$$dV=\mathcal{T}(ds-dx)=\frac{\mathcal{T}}{2}\left(\frac{d\psi}{dx}\right)^2dx \tag{3}$$

We can now introduce the *Lagrangian density* (effectively, the Lagrangian per unit length):

$$\mathcal{L}\equiv\frac{dL}{dx}=\frac{dT-dV}{dx}=\frac{\rho}{2}(\partial_t\psi)^2-\frac{\mathcal{T}}{2}(\partial_x\psi)^2 \tag{4}$$

The action can now be written as

$$S=\int\mathcal{L}dxdt \tag{5}$$

To minimize the action we now have to consider a functional that depends on a function of two independent variables. That is, we have

$$\mathcal{L}=\mathcal{L}\left(\psi,\partial_x\psi,\partial_t\psi\right) \tag{6}$$

where

$$\psi=\psi\left(x,t\right) \tag{7}$$

We can generalize the original definition of a functional derivative by considering a variation of the function ψ at one point (x_0,t_0), that is

$$\psi\left(x,t\right)\to\psi\left(x,t\right)+\epsilon\delta\left(x-x_0\right)\delta\left(t-t_0\right) \tag{8}$$

To see how this works, consider the slightly simpler case where the function $\mathcal{L}$ depends only on ψ and not on its derivatives. Then

$$\frac{\delta S\left[\psi\left(x,t\right)\right]}{\delta\psi\left(x_0,t_0\right)}=\lim_{\epsilon\to0}\frac{1}{\epsilon}\int dt\int dx\left[\mathcal{L}\left(\psi+\epsilon\delta\left(x-x_0\right)\delta\left(t-t_0\right)\right)-\mathcal{L}\left(\psi\right)\right] \tag{9}$$

$$=\int dt\int dx\frac{\partial\mathcal{L}}{\partial\psi\left(x,t\right)}\delta\left(x-x_0\right)\delta\left(t-t_0\right) \tag{10}$$

$$=\int dt\frac{\partial\mathcal{L}}{\partial\psi\left(x_0,t\right)}\delta\left(t-t_0\right) \tag{11}$$

$$=\frac{\partial\mathcal{L}}{\partial\psi\left(x_0,t_0\right)} \tag{12}$$

where we used a Taylor expansion of the first line up to first order to get the second line.

To extend this to the more general case where $\mathcal{L}=\mathcal{L}\left(\psi,\partial_x\psi,\partial_t\psi\right)$ we can follow example 2 in this earlier post to get

$$\frac{\delta S\left[\psi\left(x,t\right)\right]}{\delta\psi\left(x_0,t_0\right)}=\frac{\partial\mathcal{L}}{\partial\psi\left(x_0,t_0\right)}-\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\left(\partial_x\psi\left(x_0,t_0\right)\right)}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi\left(x_0,t_0\right)\right)}=0 \tag{13}$$

Applying this to 4 we get

$$\frac{\partial\mathcal{L}}{\partial\psi\left(x_0,t_0\right)}-\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\left(\partial_x\psi\left(x_0,t_0\right)\right)}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi\left(x_0,t_0\right)\right)}=0+\mathcal{T}\frac{d}{dx}\left(\partial_x\psi\right)-\rho\frac{d}{dt}\left(\partial_t\psi\right)=0 \tag{14}$$

$$\partial_x^2\psi=\frac{\rho}{\mathcal{T}}\partial_t^2\psi \tag{15}$$

The final equation is just the wave equation that we derived earlier when studying electromagnetic waves.

We can generalize this to a three-dimensionsal elastic medium by taking $\psi(x,y,z,t)=\psi(\mathbf{r},t)$ to be a three-dimensional scalar field (actually, it's not clear exactly what ψ represents in this case; in the 'real' theory of 3-d elasticity, the displacement of an element of the elastic medium is a vector field, not a scalar field, as you would expect). We can generalize the idea of a Lagrangian density to three dimensions, so we have

$$\mathcal{L}=\frac{\rho}{2}(\partial_t\psi)^2-\frac{\mathcal{T}}{2}(\nabla\psi)^2 \tag{16}$$

The principle of least action is now

$$\frac{\delta S[\psi(\mathbf{r},t)]}{\delta\psi(\mathbf{r}_0,t_0)}=\frac{\delta}{\delta\psi(\mathbf{r}_0,t_0)}\int dt\int d^3\mathbf{r}\left[\frac{\rho}{2}(\partial_t\psi)^2-\frac{\mathcal{T}}{2}(\nabla\psi)^2\right] \tag{17}$$

$$=\frac{\delta}{\delta\psi(\mathbf{r}_0,t_0)}\int dt\int d^3\mathbf{r}\left[\frac{\rho}{2}(\partial_t\psi)^2-\frac{\mathcal{T}}{2}\left[(\partial_x\psi)^2+(\partial_y\psi)^2+(\partial_z\psi)^2\right]\right] \tag{18}$$

ψ is now a function of 4 independent variables, so we can use the same argument as above to calculate the functional derivative. We perturb ψ by the amount $\epsilon\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t-t_0)$ and follow through the derivation in the same way. All the terms in the integral are of the form we looked at in example 4 in this post, so we get

$$\frac{\delta S[\psi(\mathbf{r},t)]}{\delta\psi(\mathbf{r}_0,t_0)}=-\rho\partial_t^2\psi+\mathcal{T}\left(\partial_x^2\psi+\partial_y^2\psi+\partial_z^2\psi\right)=0 \tag{19}$$

$$\nabla^2\psi=\frac{\rho}{\mathcal{T}}\partial_t^2\psi \tag{20}$$

We get the 3-d wave equation.

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