

FUNCTIONAL DERIVATIVE: A 4-DIMENSIONAL EXAMPLE

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 1.6.

Now for a more involved example of a functional derivative. We define a functional as

$$(1) \quad Z_0[J] = \exp \left[-\frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right]$$

where $\Delta(x) = \Delta(-x)$.

Here, we're taking x and y to be four-dimensional vectors (such as might be used in relativity to represent space-time). To find the functional derivative $\frac{\delta Z_0[J]}{\delta J(z)}$ we can use the generalization of the functional derivative that we defined earlier. That is, we have

$$(2) \quad \frac{\delta Z_0[J]}{\delta J(z)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(Z_0 \left[J(x) + \epsilon \delta^{(4)}(z-x) \right] - Z_0[J(x)] \right)$$

where $\delta^{(4)}$ is the four-dimensional delta function

$$(3) \quad \delta^{(4)}(x) = \delta(x_1) \delta(x_2) \delta(x_3) \delta(x_4)$$

To find the derivative, we first find $Z_0 \left[J(x) + \epsilon \delta^{(4)}(z-x) \right]$ to first order in ϵ .

$$\begin{aligned}
(4) \quad Z_0 [J(x) + \varepsilon \delta^{(4)}(z-x)] &= \exp \left[-\frac{1}{2} \int d^4x d^4y \left(J(x) + \varepsilon \delta^{(4)}(z-x) \right) \times \right. \\
(5) \quad &\quad \left. \Delta(x-y) \left(J(y) + \varepsilon \delta^{(4)}(z-y) \right) \right] \\
(6) \quad &\approx \exp \left[-\frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right] \times \\
(7) \quad &\exp \left[-\frac{\varepsilon}{2} \int d^4x d^4y \left(\delta^{(4)}(z-x) \Delta(x-y) J(y) + \delta^{(4)}(z-y) \Delta(x-y) J(x) \right) \right] \\
(8) \quad &= Z_0[J] \exp \left[-\frac{\varepsilon}{2} \int d^4x d^4y \left(\delta^{(4)}(z-x) \Delta(x-y) J(y) + \right. \right. \\
(9) \quad &\quad \left. \left. \delta^{(4)}(z-y) \Delta(x-y) J(x) \right) \right] \\
(10) \quad &= Z_0[J] \exp \left[-\frac{\varepsilon}{2} \left(\int d^4y \Delta(z-y) J(y) + \int d^4x \Delta(x-z) J(x) \right) \right] \\
(11) \quad &= Z_0[J] \exp \left[-\varepsilon \int d^4y \Delta(z-y) J(y) \right]
\end{aligned}$$

where we renamed the integration variable x to y and used $\Delta(x) = \Delta(-x)$ in 10.

Since the exponent in the last line is small, we can expand it in a Taylor series and keep only up to the first order term:

$$(12) \quad Z_0[J] \exp \left[-\varepsilon \int d^4y \Delta(z-y) J(y) \right] \approx Z_0[J] \left(1 - \varepsilon \int d^4y \Delta(z-y) J(y) \right)$$

Plugging this back into 2 and taking the limit we get

$$(13) \quad \frac{\delta Z_0[J]}{\delta J(z)} = -Z_0[J] \int d^4y \Delta(z-y) J(y)$$