FUNCTIONAL DERIVATIVE: A 4-DIMENSIONAL EXAMPLE

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 1.6.

Now for a more involved example of a functional derivative. We define a functional as

$$Z_0[J] = \exp\left[-\frac{1}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right] \tag{1}$$

where $\Delta(x) = \Delta(-x)$.

Here, we're taking x and y to be four-dimensional vectors (such as might be used in relativity to represent space-time). To find the functional derivative $\frac{\delta Z_0[J]}{\delta J(z_1)}$ we can use the generalization of the functional derivativethat we defined earlier. That is, we have

$$\frac{\delta Z_0[J]}{\delta J(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right] - Z_0[J(x)]\right) \tag{2}$$

where $\delta^{(4)}$ is the four-dimensional delta function

$$\delta^{(4)}(x) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4) \tag{3}$$

To find the derivative, we first find $Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right]$ to first order in ϵ.

$$Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right]=\exp\left[-\frac{1}{2}\int d^4xd^4y\left(J(x)+\epsilon\delta^{(4)}(z-x)\right)\times\right. \tag{4}$$

$$\left.\Delta(x-y)\left(J(y)+\epsilon\delta^{(4)}(z-y)\right)\right] \tag{5}$$

$$\approx\exp\left[-\frac{1}{2}\int d^4xd^4yJ(x)\Delta(x-y)J(y)\right]\times \tag{6}$$

$$\exp\left[-\frac{\epsilon}{2}\int d^4xd^4y\left(\delta^{(4)}(z-x)\Delta(x-y)J(y)+\delta^{(4)}(z-y)\Delta(x-y)J(x)\right)\right] \tag{7}$$

$$=Z_0[J]\exp\left[-\frac{\epsilon}{2}\int d^4xd^4y\left(\delta^{(4)}(z-x)\Delta(x-y)J(y)+\right.\right. \tag{8}$$

$$\left.\left.\delta^{(4)}(z-y)\Delta(x-y)J(x)\right)\right] \tag{9}$$

$$=Z_0[J]\exp\left[-\frac{\epsilon}{2}\left(\int d^4y\Delta(z-y)J(y)+\int d^4x\Delta(x-z)J(x)\right)\right] \tag{10}$$

$$=Z_0[J]\exp\left[-\epsilon\int d^4y\Delta(z-y)J(y)\right] \tag{11}$$

where we renamed the integration variable x to y and used $\Delta(x)=\Delta(-x)$ in 10.

Since the exponent in the last line is small, we can expand it in a Taylor series and keep only up to the first order term:

$$Z_0[J]\exp\left[-\epsilon\int d^4y\Delta(z-y)J(y)\right]\approx Z_0[J]\left(1-\epsilon\int d^4y\Delta(z-y)J(y)\right) \tag{12}$$

Plugging this back into 2 and taking the limit we get

$$\frac{\delta Z_0[J]}{\delta J(z)}=-Z_0[J]\int d^4y\Delta(z-y)J(y) \tag{13}$$