CREATION AND ANNIHILATION OPERATORS IN THE HARMONIC OSCILLATOR

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problems 2.1,2.2.

We've already seen a hint of quantum field theory when we studied the harmonic oscillator in non-relativistic quantum theory. We used raising and lowering operators to move between energy levels in an oscillator. We'll quote the earlier results, but change the notation so that it matches that used in Lancaster & Blundell, where the raising operator is $\hat{a}^{\dagger}$ and the lowering operator is $\hat{a}$.

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}\left[-i\hat{p}+m\omega\hat{x}\right] \tag{1}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left[i\hat{p}+m\omega\hat{x}\right] \tag{2}$$

Solving for $\hat{x}$ and $\hat{p}$ we get

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^{\dagger}+\hat{a}\right) \tag{3}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^{\dagger}-\hat{a}\right) \tag{4}$$

We also showed that the oscillator hamiltonian can be written in terms of these operators as

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right) \tag{5}$$

The operators have the commutator

$$\left[\hat{a},\hat{a}^{\dagger}\right] = 1 \tag{6}$$

Both operators also commute with themselves, since all operators commute with themselves.

We also showed that normalization of these operators gives the formulas:

$$\begin{aligned} \hat{a}^{\dagger}|n\rangle &= \sqrt{n+1}\,|n+1\rangle & (7)\\ \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle & (8)\end{aligned}$$

When viewed from quantum field theory, the raising operator is seen as an operator that creates a particle of energy $\hbar\omega$ and is therefore called a *creation operator*. Similarly, the lowering operator annihilates a particle of energy $\hbar\omega$ and is called an *annihilation operator*. That is, the quantum of energy equal to the difference in adjacent levels in a harmonic oscillator is viewed as a particle.

We can use creation and annihilation operators together with perturbation theory to calculate energy levels of perturbed oscillators. For example, suppose we have a hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4 \tag{9}$$

where λ is small. The first two terms are the unperturbed harmonic oscillator, so we can take the perturbation to be

$$H' = \lambda\hat{x}^4 \tag{10}$$

Then the perturbation in energy in the state$|n\rangle$ of the original unperturbed hamiltonian is given by the matrix element of the perturbation:

$$E'_n = \langle n\,|H'|\,n\rangle \tag{11}$$

Expressing H' in terms of $\hat{a}^{\dagger}$ and $\hat{a}$ via 3 we get

$$H' = \lambda\left(\frac{\hbar}{2m\omega}\right)^2\left(\hat{a}^{\dagger}+\hat{a}\right)^4 \tag{12}$$

Because we're interested only in terms in this operator that leave the number of particles unchanged (since we're calculating $\langle n\,|H'|\,n\rangle$ and the states of the original hamiltonian are orthogonal) we can keep only those terms in the expansion of $\left(\hat{a}^{\dagger}+\hat{a}\right)^4$ that contain two of each operator. That is, we have

$$\langle n\,|H'|\,n\rangle = \left\langle n\left|\hat{a}^2\left(\hat{a}^{\dagger}\right)^2 + \hat{a}\left(\hat{a}^{\dagger}\right)^2\hat{a} + \left(\hat{a}^{\dagger}\right)^2\hat{a}^2 + \hat{a}^{\dagger}\hat{a}^2\hat{a}^{\dagger} + \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\right|n\right\rangle \tag{13}$$

Applying the equations 7 and 8 we get

$$\left\langle n\left|H'\right|n\right\rangle = (n+1)(n+2)+n(n+1)+n(n-1)+(n+1)n+(n+1)^2+n^2 \tag{14}$$

$$= 6n^2+6n+3 \tag{15}$$

so the energy levels, correct to first order in λ, are

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)+\lambda\left(\frac{\hbar}{2m\omega}\right)^2\left(6n^2+6n+3\right) \tag{16}$$

$$= \hbar\omega\left(n+\frac{1}{2}\right)+\frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2\left(2n^2+2n+1\right) \tag{17}$$

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