HARMONIC OSCILLATOR GROUND STATE FROM ANNIHILATION OPERATOR

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 2.4.

We can use the annihilation operator $\hat{a}$ in the harmonic oscillator to reclaim the position space form of the ground state wave function. The operator is

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left[i\hat{p}+m\omega\hat{x}\right] \tag{1}$$

Applying $\hat{a}$ to the ground state $|0\rangle$ we get 0 (that is, annihilating the ground state eliminates the wave function altogether), so

$$\left[i\hat{p}+m\omega\hat{x}\right]|0\rangle = 0 \tag{2}$$

The eigenfunction of position is found from

$$\hat{x}|x_0\rangle = x_0|x_0\rangle \tag{3}$$

Since the operator $\hat{x}$ multiplies any function by the position x and we want the eigenfunction $|x_0\rangle$ to represent a particular position x_0, $|x_0\rangle$ must pick out x_0 from all possible values of x, that is, it must be zero everywhere except $x = x_0$. This condition is satisfied if we take

$$|x_0\rangle = \delta(x-x_0) \tag{4}$$

We then get

$$\langle x|\hat{p}|\psi\rangle = \int \delta\left(x'-x\right)\hat{p}\psi\left(x'\right)\,dx' \tag{5}$$

$$= -i\hbar\int \delta\left(x'-x\right)\frac{d}{dx'}\psi\left(x'\right)\,dx' \tag{6}$$

$$= -i\hbar\frac{d}{dx}\int \delta\left(x'-x\right)\psi\left(x'\right)\,dx' \tag{7}$$

$$= -i\hbar\frac{d}{dx}\langle x|\psi\rangle \tag{8}$$

Also

$$\langle x|\hat{x}|\psi\rangle = \int \delta\left(x'-x\right)\hat{x}\psi\left(x'\right)\,dx' \tag{9}$$

$$= \int \delta\left(x'-x\right)x'\psi\left(x'\right)\,dx' \tag{10}$$

$$= x\int \delta\left(x'-x\right)\psi\left(x'\right)\,dx' \tag{11}$$

$$= x\langle x|\psi\rangle \tag{12}$$

Therefore, from 2 we get

$$\langle x|[i\hat{p}+m\omega\hat{x}]|0\rangle = \hbar\frac{d}{dx}\langle x|0\rangle + m\omega x\langle x|0\rangle = 0 \tag{13}$$

$$\hbar\frac{d}{dx}\langle x|0\rangle = -m\omega x\langle x|0\rangle \tag{14}$$

This is a differential equation for $\langle x|0\rangle$ which has the solution

$$\langle x|0\rangle = Ae^{-m\omega x^2/2\hbar} \tag{15}$$

where A is found from normalization:

$$\int |\langle x|0\rangle|^2\,dx = A^2\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}dx = 1 \tag{16}$$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{17}$$

This is the same function that we got earlier.