

HARMONIC OSCILLATOR GROUND STATE FROM ANNIHILATION OPERATOR

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 2.4.

We can use the annihilation operator \hat{a} in the harmonic oscillator to reclaim the position space form of the ground state wave function. The operator is

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}} [i\hat{p} + m\omega\hat{x}] \quad (1)$$

Applying \hat{a} to the ground state $|0\rangle$ we get 0 (that is, annihilating the ground state eliminates the wave function altogether), so

$$[i\hat{p} + m\omega\hat{x}] |0\rangle = 0 \quad (2)$$

The eigenfunction of position is found from

$$\hat{x} |x_0\rangle = x_0 |x_0\rangle \quad (3)$$

Since the operator \hat{x} multiplies any function by the position x and we want the eigenfunction $|x_0\rangle$ to represent a particular position x_0 , $|x_0\rangle$ must pick out x_0 from all possible values of x , that is, it must be zero everywhere except $x = x_0$. This condition is satisfied if we take

$$|x_0\rangle = \delta(x - x_0) \quad (4)$$

We then get

$$\langle x | \hat{p} | \psi \rangle = \int \delta(x' - x) \hat{p} \psi(x') dx' \quad (5)$$

$$= -i\hbar \int \delta(x' - x) \frac{d}{dx'} \psi(x') dx' \quad (6)$$

$$= -i\hbar \frac{d}{dx} \int \delta(x' - x) \psi(x') dx' \quad (7)$$

$$= -i\hbar \frac{d}{dx} \langle x | \psi \rangle \quad (8)$$

Also

$$\langle x | \hat{x} | \psi \rangle = \int \delta(x' - x) \hat{x} \psi(x') dx' \quad (9)$$

$$= \int \delta(x' - x) x' \psi(x') dx' \quad (10)$$

$$= x \int \delta(x' - x) \psi(x') dx' \quad (11)$$

$$= x \langle x | \psi \rangle \quad (12)$$

Therefore, from 2 we get

$$\langle x | [i\hat{p} + m\omega\hat{x}] | 0 \rangle = \hbar \frac{d}{dx} \langle x | 0 \rangle + m\omega x \langle x | 0 \rangle = 0 \quad (13)$$

$$\hbar \frac{d}{dx} \langle x | 0 \rangle = -m\omega x \langle x | 0 \rangle \quad (14)$$

This is a differential equation for $\langle x | 0 \rangle$ which has the solution

$$\langle x | 0 \rangle = A e^{-m\omega x^2 / 2\hbar} \quad (15)$$

where A is found from normalization:

$$\int |\langle x | 0 \rangle|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-m\omega x^2 / \hbar} dx = 1 \quad (16)$$

$$A = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \quad (17)$$

This is the same function that we got earlier.