HARMONIC OSCILLATOR GROUND STATE FROM ANNIHILATION OPERATOR

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 2.4.

We can use the annihilation operator $\hat{a}$ in the harmonic oscillator to reclaim the position space form of the ground state wave function. The operator is

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left[i\hat{p}+m\omega\hat{x}\right] \tag{1}$$

Applying $\hat{a}$ to the ground state $|0\rangle$ we get 0 (that is, annihilating the ground state eliminates the wave function altogether), so

$$\left[i\hat{p}+m\omega\hat{x}\right]|0\rangle = 0 \tag{2}$$

The eigenfunction of position is found from

$$\hat{x}|x_0\rangle = x_0|x_0\rangle \tag{3}$$

Since the operator $\hat{x}$ multiplies any function by the position x and we want the eigenfunction $|x_0\rangle$ to represent a particular position x_0, $|x_0\rangle$ must pick out x_0 from all possible values of x, that is, it must be zero everywhere except $x = x_0$. This condition is satisfied if we take

$$|x_0\rangle = \delta(x-x_0) \tag{4}$$

We then get

$$\begin{aligned}
\langle x|\hat{p}|\psi\rangle &= \int \delta\left(x'-x\right)\hat{p}\psi\left(x'\right)\,dx' &(5)\\
&= -i\hbar\int \delta\left(x'-x\right)\frac{d}{dx'}\psi\left(x'\right)\,dx' &(6)\\
&= -i\hbar\frac{d}{dx}\int \delta\left(x'-x\right)\psi\left(x'\right)\,dx' &(7)\\
&= -i\hbar\frac{d}{dx}\langle x|\psi\rangle &(8)
\end{aligned}$$

Also

$$\begin{aligned}
\langle x|\hat{x}|\psi\rangle &= \int\delta\left(x'-x\right)\hat{x}\psi\left(x'\right)\,dx' && (9)\\
&= \int\delta\left(x'-x\right)x'\psi\left(x'\right)\,dx' && (10)\\
&= x\int\delta\left(x'-x\right)\psi\left(x'\right)\,dx' && (11)\\
&= x\langle x|\psi\rangle && (12)
\end{aligned}$$

Therefore, from 2 we get

$$\begin{aligned}
\langle x|[i\hat{p}+m\omega\hat{x}]|0\rangle &= \hbar\frac{d}{dx}\langle x|0\rangle+m\omega x\langle x|0\rangle=0 && (13)\\
\hbar\frac{d}{dx}\langle x|0\rangle &= -m\omega x\langle x|0\rangle && (14)
\end{aligned}$$

This is a differential equation for $\langle x|0\rangle$ which has the solution

$$\langle x|0\rangle=Ae^{-m\omega x^{2}/2\hbar} \tag{15}$$

where A is found from normalization:

$$\begin{aligned}
\int|\langle x|0\rangle|^{2}dx &= A^{2}\int_{-\infty}^{\infty}e^{-m\omega x^{2}/\hbar}dx=1 && (16)\\
A &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} && (17)
\end{aligned}$$

This is the same function that we got earlier.