OCCUPATION NUMBER REPRESENTATION; DELTA FUNCTION AS A SERIES

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References: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014) - Problem 3.1.

We can write the hamiltonian for the harmonic oscillator in terms of the creation and annihilation operators as

$$\hat{H}=\hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right) \tag{1}$$

Normalization requires

$$\begin{aligned} a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle &(2)\\ a^{\dagger}\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle &(3)\end{aligned}$$

so the combined operator $a^{\dagger}a$ acts as a number operator, giving the number of quanta in a state:

$$\begin{aligned} a^{\dagger}a\left|n\right\rangle &= a^{\dagger}\sqrt{n}\left|n-1\right\rangle &(4)\\ &= \sqrt{n}a^{\dagger}\left|n-1\right\rangle &(5)\\ &= n\left|n\right\rangle &(6)\end{aligned}$$

We can generalize this to a collection of independent oscillators where oscillator k has frequency ω_k. In that case

$$\hat{H}=\hbar\sum_{k}\omega_{k}\left(a_{k}^{\dagger}a_{k}+\frac{1}{2}\right) \tag{7}$$

where $a_k^{\dagger}$ and a_k are the creation and annihilation operators for one quantum in oscillator k. For the harmonic oscillator, the energy levels are all equally spaced, with a spacing of $\hbar\omega_k$ so if we redefine the zero point of energy to be $\frac{1}{2}\hbar\omega_k$ for oscillator k, then the hamiltonian above can be rewritten as

$$\hat{H}=\sum_{k}n_{k}\hbar\omega_{k} \tag{8}$$

where n_k is the number of quanta in oscillator k. An eigenstate of this hamiltonian is a state containing N oscillators with oscillator k containing n_k quanta, which we can write as $|n_1 n_2 \ldots n_N\rangle$. This is called the *occupation number representation* since rather than writing out a complex wave function describing all N oscillators, we just list the number of quanta contained within each oscillator.

The application of this to quantum field theory is that we can interpret each quantum in oscillator k as a particle with a momentum p_k. We're not saying that a particle *is* an oscillator; rather we're noting that we can use the same notation to refer to both particles and oscillators. So if we have a number of momentum states p_k available in our system, then we can define creation and annihilation operators $a^\dagger_{p_k}$ and a_{p_k} for that momentum state and write the hamiltonian as

$$\hat{H} = \sum_k E_{p_k} a^\dagger_{p_k} a_{p_k} \tag{9}$$

In order for creation operators to work properly when creating elementary particles, we need to recall that there are two fundamental types of particles: fermions and bosons. The wave function for two bosons is, in position space:

$$\psi(\mathbf{r}_a, \mathbf{r}_b) = A\left[\psi_1(\mathbf{r}_a)\,\psi_2(\mathbf{r}_b) + \psi_2(\mathbf{r}_a)\,\psi_1(\mathbf{r}_b)\right] \tag{10}$$

If we interchange the two particles by swapping $\mathbf{r}_a$ and $\mathbf{r}_b$, the compound wave function $\psi(\mathbf{r}_a, \mathbf{r}_b)$ doesn't change, so that $\psi(\mathbf{r}_a, \mathbf{r}_b) = \psi(\mathbf{r}_b, \mathbf{r}_a)$

If we have two fermions, on the other hand, the wave function is

$$\psi(\mathbf{r}_a, \mathbf{r}_b) = A\left[\psi_1(\mathbf{r}_a)\,\psi_2(\mathbf{r}_b) - \psi_2(\mathbf{r}_a)\,\psi_1(\mathbf{r}_b)\right] \tag{11}$$

and now if we swap the particles we get $\psi(\mathbf{r}_a, \mathbf{r}_b) = -\psi(\mathbf{r}_b, \mathbf{r}_a)$.

If we use two creation operators operating on the vacuum state $|0\rangle$ to create a state containing two particles, the resulting state must behave properly under the exchange of the two particles. Another way of putting this is that if we swap the order in which the particles are created we must get exactly the same state if the particles are bosons, but the negative of the original state if the particles are fermions. That is, for bosons

$$a^\dagger_{p_1} a^\dagger_{p_2} = a^\dagger_{p_2} a^\dagger_{p_1} \tag{12}$$

or in terms of commutators

$$\left[a^\dagger_{p_1}, a^\dagger_{p_2}\right] = 0 \tag{13}$$

For fermions, we'll use the symbols $c^\dagger$ and c for creation and annihilation operators, and in this case we must have

$$c_{p_1}^\dagger c_{p_2}^\dagger = -c_{p_2}^\dagger c_{p_1}^\dagger \tag{14}$$

For fermions we define an *anticommutator* as

$$\left\{c_{p_1}^\dagger, c_{p_2}^\dagger\right\} \equiv c_{p_1}^\dagger c_{p_2}^\dagger + c_{p_2}^\dagger c_{p_1}^\dagger \tag{15}$$

so we have

$$\left\{c_{p_1}^\dagger, c_{p_2}^\dagger\right\} = 0 \tag{16}$$

For the harmonic oscillator, the creation and annihilation operators satisfied the commutation relation

$$\left[a_{p_1}, a_{p_2}^\dagger\right] = \delta_{p_1 p_2} \tag{17}$$

That is, the annihilation operator commutes with the creation operator if they refer to different oscillators; otherwise the commutator is 1. To complete the analogy between particles and oscillators, we just *define* the commutation relations between creation and annihilation operators for particles as

$$\begin{aligned} \left[a_{p_1}, a_{p_2}^\dagger\right] &= \delta_{p_1 p_2} & (18) \\ \left\{c_{p_1}, c_{p_2}^\dagger\right\} &= \delta_{p_1 p_2} & (19) \end{aligned}$$

Example. The commutation relations can be inserted into a formula which gives a new form of the Dirac delta function. For two different momentum states $\mathbf{p}$ and $\mathbf{q}$ we have, for a pair of bosons

$$\left[a_p, a_q^\dagger\right] = \delta_{pq} \tag{20}$$

Suppose that the system is enclosed in a cube of side length L. Then we can construct the sum

$$\frac{1}{\mathcal{V}} \sum_{p,q} e^{i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \left[a_p, a_q^\dagger\right] = \frac{1}{\mathcal{V}} \sum_{p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{21}$$

What can we make of the sum on the RHS? To see what it is, suppose we have some function $f(x)$ defined for $-\pi \le x \le \pi$. We can expand it in a Fourier series as follows:

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx} \tag{22}$$

where the coefficients are

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,e^{-inx}dx \tag{23}$$

We can write the Fourier series for the function at a particular point $x=a$ as

$$\begin{aligned} f(a) &= \frac{1}{2\pi}\sum_n e^{ina}\int_{-\pi}^{\pi}f(x)\,e^{-inx}dx && (24)\\ &= \int_{-\pi}^{\pi}f(x)\left[\frac{1}{2\pi}\sum_n e^{in(a-x)}\right]dx && (25)\end{aligned}$$

The term in brackets in the last line behaves exactly like $\delta(x-a)$ so we can take it as another definition of the Dirac delta function

$$\delta(x-a)=\frac{1}{2\pi}\sum_n e^{in(a-x)}=\frac{1}{2\pi}\sum_n e^{in(x-a)} \tag{26}$$

where we can change the exponent in the last term because the sum over n extends from $-\infty$ to ∞ so we can replace n by $-n$ and get the same sum.

Now if the function $f(x)$ extends from 0 to L instead of from $-\pi$ to π we can replace x by $\xi\equiv Lx/2\pi$ (and a by $\xi_a\equiv La/2\pi$) to get

$$\begin{aligned} f(\xi_a) &= \int_0^L f(\xi)\left[\frac{1}{2\pi}\frac{2\pi}{L}\sum_n e^{i2\pi n(\xi_a-\xi)/L}\right]d\xi && (27)\\ &= \int_0^L f(\xi)\left[\frac{1}{L}\sum_p e^{ip(\xi-\xi_a)}\right]d\xi && (28)\end{aligned}$$

where

$$p\equiv\frac{2\pi n}{L} \tag{29}$$

Obviously, the same argument works for the y and z directions, so in 3-d

$$f(\mathbf{a}) = \int_{\mathcal{V}} f(\mathbf{r}) \left[\frac{1}{L^3}\sum_p e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{a})}\right] d^3\mathbf{r} \tag{30}$$

$$= \int_{\mathcal{V}} f(\mathbf{r}) \left[\frac{1}{\mathcal{V}}\sum_p e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{a})}\right] d^3\mathbf{r} \tag{31}$$

so the 3-d delta function is

$$\delta^{(3)}(\mathbf{x}-\mathbf{y}) = \frac{1}{\mathcal{V}}\sum_p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{32}$$

From 21 we get

$$\frac{1}{\mathcal{V}}\sum_{p,q} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[a_p, a_q^\dagger\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{33}$$

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