CREATION AND ANNIHILATION OPERATORS IN THE HARMONIC OSCILLATOR: A FEW THEOREMS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 3.2.

For the harmonic oscillator, we've seen that the effects of the creation and annihilation (raising and lowering) operators are

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{1}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{2}$$

Therefore

$$\left\langle m\left|\hat{a}^{\dagger}\right|n\right\rangle = \sqrt{n+1}\langle m|n+1\rangle \tag{3}$$

$$= \sqrt{n+1}\delta_{m,n+1} \tag{4}$$

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle \tag{5}$$

$$= \sqrt{n}\delta_{m,n-1} \tag{6}$$

From the commutation relation

$$\left[\hat{a},\hat{a}^{\dagger}\right] = 1 \tag{7}$$

we'd like to prove that

$$\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right] = n\left(\hat{a}^{\dagger}\right)^{n-1} \tag{8}$$

We can prove this using mathematical induction. The formula is true for $n=1$, so we assume it's true for $n-1$ and then show it's true for n.

We get

$$\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right] = \hat{a}\left(\hat{a}^{\dagger}\right)^{n}-\left(\hat{a}^{\dagger}\right)^{n}\hat{a} \tag{9}$$

$$= \left(1+\hat{a}^{\dagger}\hat{a}\right)\left(\hat{a}^{\dagger}\right)^{n-1}-\left(\hat{a}^{\dagger}\right)^{n}\hat{a} \tag{10}$$

$$= \left(\hat{a}^{\dagger}\right)^{n-1}+\hat{a}^{\dagger}\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n-1}\right] \tag{11}$$

We can now use our assumption that $\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n-1}\right]=(n-1)\left(\hat{a}^{\dagger}\right)^{n-2}$ and we get

$$\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right] = \left(\hat{a}^{\dagger}\right)^{n-1}+\hat{a}^{\dagger}(n-1)\left(\hat{a}^{\dagger}\right)^{n-2} \tag{12}$$

$$= n\left(\hat{a}^{\dagger}\right)^{n-1} \tag{13}$$

QED.

We can also work out

$$\left\langle 0\left|\hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n}\right|0\right\rangle = \sqrt{(1!)}\left\langle 0\left|\hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n-1}\right|1\right\rangle \tag{14}$$

$$= \sqrt{(2!)}\left\langle 0\left|\hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n-2}\right|2\right\rangle \tag{15}$$

$$= \ldots \tag{16}$$

$$= \sqrt{n!}\left\langle 0\left|\hat{a}^{m}\right|n\right\rangle \tag{17}$$

$$= n\sqrt{(n-1)!}\left\langle 0\left|\hat{a}^{m-1}\right|n-1\right\rangle \tag{18}$$

$$= \ldots \tag{19}$$

If $m=n$, the sequence of annihilation operators will eventually reduce the right-hand state to $|0\rangle$ with a factor of $n!$ out front. If $m<n$ then $\hat{a}^{m}$ will reduce the right-hand state to $|n-m\rangle$ and $\langle 0|n-m\rangle=\delta_{nm}=0$. If $m>n$, then after applying $\hat{a}^{n}$ we'll still have $\hat{a}^{m-n}$ left over so we'll have an annihilation operator acting on $|0\rangle$ which gives zero. Therefore

$$\left\langle 0\left|\hat{a}^{m}\left(\hat{a}^{\dagger}\right)^{n}\right|0\right\rangle = n!\delta_{nm} \tag{20}$$