

CREATION AND ANNIHILATION OPERATORS IN THE HARMONIC OSCILLATOR: A FEW THEOREMS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 3.2.

For the harmonic oscillator, we've seen that the effects of the creation and annihilation (raising and lowering) operators are

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (1)$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (2)$$

Therefore

$$\langle m | \hat{a}^\dagger | n \rangle = \sqrt{n+1} \langle m | n+1 \rangle \quad (3)$$

$$= \sqrt{n+1} \delta_{m,n+1} \quad (4)$$

$$\langle m | \hat{a} | n \rangle = \sqrt{n} \langle m | n-1 \rangle \quad (5)$$

$$= \sqrt{n} \delta_{m,n-1} \quad (6)$$

From the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (7)$$

we'd like to prove that

$$[\hat{a}, (\hat{a}^\dagger)^n] = n (\hat{a}^\dagger)^{n-1} \quad (8)$$

We can prove this using mathematical induction. The formula is true for $n = 1$, so we assume it's true for $n - 1$ and then show it's true for n .

We get

$$[\hat{a}, (\hat{a}^\dagger)^n] = \hat{a} (\hat{a}^\dagger)^n - (\hat{a}^\dagger)^n \hat{a} \quad (9)$$

$$= (1 + \hat{a}^\dagger \hat{a}) (\hat{a}^\dagger)^{n-1} - (\hat{a}^\dagger)^n \hat{a} \quad (10)$$

$$= (\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger [\hat{a}, (\hat{a}^\dagger)^{n-1}] \quad (11)$$

We can now use our assumption that $[\hat{a}, (\hat{a}^\dagger)^{n-1}] = (n-1)(\hat{a}^\dagger)^{n-2}$ and we get

$$[\hat{a}, (\hat{a}^\dagger)^n] = (\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger (n-1) (\hat{a}^\dagger)^{n-2} \quad (12)$$

$$= n (\hat{a}^\dagger)^{n-1} \quad (13)$$

QED.

We can also work out

$$\langle 0 | \hat{a}^m (\hat{a}^\dagger)^n | 0 \rangle = \sqrt{(1!)} \langle 0 | \hat{a}^m (\hat{a}^\dagger)^{n-1} | 1 \rangle \quad (14)$$

$$= \sqrt{(2!)} \langle 0 | \hat{a}^m (\hat{a}^\dagger)^{n-2} | 2 \rangle \quad (15)$$

$$= \dots \quad (16)$$

$$= \sqrt{n!} \langle 0 | \hat{a}^m | n \rangle \quad (17)$$

$$= n \sqrt{(n-1)!} \langle 0 | \hat{a}^{m-1} | n-1 \rangle \quad (18)$$

$$= \dots \quad (19)$$

If $m = n$, the sequence of annihilation operators will eventually reduce the right-hand state to $|0\rangle$ with a factor of $n!$ out front. If $m < n$ then \hat{a}^m will reduce the right-hand state to $|n-m\rangle$ and $\langle 0 | n-m \rangle = \delta_{nm} = 0$. If $m > n$, then after applying \hat{a}^n we'll still have \hat{a}^{m-n} left over so we'll have an annihilation operator acting on $|0\rangle$ which gives zero. Therefore

$$\langle 0 | \hat{a}^m (\hat{a}^\dagger)^n | 0 \rangle = n! \delta_{nm} \quad (20)$$